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Regarding the Reaction $\overline{\mathrm{K}} + \mathrm{p} \to \Lambda^0 + 2\pi$ (*).

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Summary. — A recent proposal to determine $P_{\mathrm{K}\Lambda}$, the relative KΛ parity, by a study of the reaction $\overline{\mathrm{K}} + \mathrm{p} \to \Lambda^0 + 2\pi$, in flight and at low energies, is generalized to include the admittedly remote possibility of a spin 2 K-meson. The appropriate forms of the differential cross-sections are found, the treatment being non-relativistic and strictly phenomenological. The approximation for the matrix elements is based on barrier penetration considerations. Final state interactions are neglected. If the assumptions, which are the same as those made previously, are valid, $P_{\mathrm{K}\Lambda}$ may be determined independently of whether the K-meson spin is zero or 2. In addition, there is a possibility of distinguishing spin zero from spin 2.

1. – Introduction.

It has recently been suggested that a study of the absorption of low energy K^- mesons by hydrogen according to the infrequently occurring reactions

$$\mathrm{K}^- + \mathrm{p} \to \Lambda^0 + \pi^0 + \pi^0 \tag{1}$$

$$\mathrm{K}^- + \mathrm{p} \to \Lambda^0 + \pi^+ + \pi^- \tag{2}$$

might yield information regarding $P_{\mathrm{K}\Lambda}$, the KΛ relative parity. These reactions were discussed by OKUN' and POMERANČUK [1], who considered absorption

(*) Supported by the National Science Foundation.

(1) L. B. OKUN' and I. IA. POMERANČUK: *Žurn. Éksp. Teor. Fiz.*, **34**, 997 (1958).

at rest from S state Bohr orbits, and by FONDA and RUSSELL [2], who considered absorption in flight. The advantage of studying these reactions in flight lies in the fact that at present there is considerable doubt that practically all those K^- mesons which are captured by hydrogen reach the lowest Bohr orbit before being absorbed [3]. It was assumed in these proposals that the spins of the K^- meson and the Λ^0 hyperon were zero and $\frac{1}{2}$, respectively. The wavelengths of the particles involved in these reactions are large compared with the region of interaction, thus suggesting that the momentum dependence of the transition matrix elements is determined largely by the penetrability of the centrifugal barrier. It was shown that both the angular correlations of the reaction products and the energy dependence of the total cross-sections should be sensitive to the value of $P_{K\Lambda}$. Furthermore the considerations on absorption in flight apply also to the reaction

$$\overline{K}^0 + p \to \Lambda^0 + \pi^+ + \pi^0 , \tag{3}$$

and a study of this process, if possible, would be of considerable interest in view of the recent speculations of PAIS [4] regarding the possibility of the charged and neutral K mesons having different relative parities.

There now exists conclusive evidence [5], based only on the assumption that angular momentum is conserved in the decay process $\Lambda^0 \to p + \pi^-$, that the spin of the Λ^0 is $\frac{1}{2}$. It would therefore seem reasonable to ascertain how critically determinations of $P_{K\Lambda}$ by means of reactions (1), (2) and (3) would depend on assumptions regarding the K meson spin. The K mesons are commonly assumed to have equal spin, and in recent years a considerable amount of evidence has been amassed which strongly favors the spin assignment zero [6]. Although odd integer spin values appear to be definitely excluded by the ana-

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[2] L. FONDA and J. E. RUSSELL: *Phys. Rev. Lett.*, **2**, 57 (1959).

[3] J. M. BLATT and S. T. BUTLER: *Nuovo Cimento*, **3**, 409 (1956); R. GATTO: *Nuovo Cimento*, **3**, 1142 (1956).

Note added in proof. – See, however, T. B. DAY, G. A. SNOW and J. SUCHER: *Phys. Rev. Lett.*, **3**, 61 (1959).

[4] A. PAIS: *Phys. Rev.*, **112**, 624 (1958).

[5] T. D. LEE and C. N. YANG: *Phys. Rev.*, **109**, 1755 (1958); F. S. CRAWFORD, M. CRESTI, M. L. GOOD, M. L. STEVENSON and H. K. TICHO: *Phys. Rev. Lett.*, **2**, 114 (1959).

[6] See, for example, the discussion given by F. EISLER, R. PLANO, A. PRODELL, N. SAMIOS, M. SCHWARTZ, J. STEINBERGER, P. BASSI, V. BORELLI, G. PUPPI, H. TANAKA, P. WALOSCHEK, V. ZOBOLI, M. CONVERSI, P. FRANZINI, I. MANNELLI, R. SANTANGELO, V. SILVESTRINI, G. L. BROWN, D. A. GLASER and C. GRAVES: *Nuovo Cimento*, **7**, 222 (1958).

lysis (7) of τ^+ decay and also by existence (8) of the $2\pi^0$ decay mode of the 0^0, the evidence against even, non-zero spin values is less direct, the most convincing argument (9) being based on the persistent absence of the decay modes $K^\pm\to\pi^\pm+\gamma$.

It will be shown below that, *all other assumptions being valid*, analyses of reactions (1), (2), and (3) may still be used to determine $P_{K\Lambda}$ if the K meson spin is either zero or 2. Furthermore, if $P_{K\Lambda}=+1$ spin zero can be distinguished from spin 2, and if $P_{K\Lambda}=-1$ there is some possibility of making such a distinction.

2. – Qualitative considerations.

In the center of mass system the total kinetic energy W of the products of reactions (1), (2), or (3) is given non-relativistically by

$$W=\sum_{i=1}^{3} p_i^2/(2m_i)=Q+p_K^2/(2\mu)\,, \tag{4}$$

where $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are the momenta of the two pions, $\boldsymbol{p}_3$ is the momentum of the Λ^0 hyperon, and $\boldsymbol{p}_K$ and μ_K are, respectively, the relative momentum and the reduced mass of the initial $\overline{K}p$ system. The values of Q in reactions (1), (2), and (3) are 47 MeV, 38 MeV, and 47 MeV, respectively. The total angular momentum operator $\boldsymbol{L}$ for the reaction products is

$$\boldsymbol{L}=\sum_{i=1}^{3} \boldsymbol{r}_i\times\boldsymbol{p}_i\,, \tag{5}$$

$\boldsymbol{r}_i$ being the position co-ordinate conjugate to $\boldsymbol{p}_i$.

The reaction products are described more conveniently in terms of another set of variables which are related to the $\boldsymbol{r}_i$ and $\boldsymbol{p}_i$ by

$$\boldsymbol{R}=(m_1\boldsymbol{r}_1+m_2\boldsymbol{r}_2+m_3\boldsymbol{r}_3)/(m_1+m_2+m_3)$$

$$\boldsymbol{r}_\pi=\boldsymbol{r}_1-\boldsymbol{r}_2$$

$$\boldsymbol{r}_\Lambda=\boldsymbol{r}_3-(m_1\boldsymbol{r}_1+m_2\boldsymbol{r}_2)/(m_1+m_2)$$

and

$$\boldsymbol{P}=\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p}_3=0$$

$$\boldsymbol{p}_\pi=(m_2\boldsymbol{p}_1-m_1\boldsymbol{p}_2)/(m_1+m_2)$$

$$\boldsymbol{p}_\Lambda=\boldsymbol{p}_3\,.$$

(7) J. Orear: *Phys. Rev.*, **106**, 834 (1957).

(8) F. Eisler, R. Plano, N. Samios, M. Schwartz and J. Steinberger: *Nuovo Cimento*, **5**, 1700 (1957).

(9) R. H. Dalitz: *Phys. Rev.*, **99**, 915 (1955).

It is readily verified that W and $\boldsymbol{L}$ may be written as

$$W = p_\pi^2/(2\mu_\pi) + p_\Lambda^2/(2\mu_\Lambda)\,, \tag{6}$$

$$\boldsymbol{L} = \boldsymbol{l}_\pi + \boldsymbol{l}_\Lambda\,, \tag{7}$$

where

$$\mu_\pi = m_1 m_2/(m_1+m_2)\,,$$

$$\mu_\Lambda = (m_1+m_2)m_3/(m_1+m_2+m_3)\,,$$

and

$$\boldsymbol{l}_\pi = \boldsymbol{r}_\pi \times \boldsymbol{p}_\pi\,, \qquad \boldsymbol{l}_\Lambda = \boldsymbol{r}_\Lambda \times \boldsymbol{p}_\Lambda\,.$$

In a channel corresponding to the products of reactions (1), (2), or (3) the outgoing wave which is produced by a particular incident partial wave with total angular momentum J and orbital angular momentum l_K is described asymptotically by an infinite series of product wave functions representing couplings of the Λ^0 spin with all the possible combinations of l_π and l_Λ which are compatible with the possible identity of the pions and the conservation of angular momentum and parity. The wave number $k_K = p_K/\hbar$ and the maximum values of the wave numbers $k_\pi = p_\pi/\hbar$ and $k_\Lambda = p_\Lambda/\hbar$ are given by

$$\begin{cases} k_K = .13\,(W_{\text{MeV}} - Q_{\text{MeV}})^{\frac{1}{2}}\,(10)^{13}\ \text{cm}^{-1} \\ k_{\pi,\,\max} = .06\,(W_{\text{MeV}})^{\frac{1}{2}}\,(10)^{13}\ \text{cm}^{-1} \\ k_{\Lambda,\max} = .11\,(W_{\text{MeV}})^{\frac{1}{2}}\,(10)^{13}\ \text{cm}^{-1}\,, \end{cases} \tag{8}$$

where W_{MeV} and Q_{MeV} are the values of W and Q in MeV. The usual theoretical estimate of the range of the $\overline{\text{K}}\mathcal{N}$ interaction is $\sim \hbar/(M_K c) = .4\cdot 10^{-13}$ cm. Therefore, at low energies the effects of high orbital angular momenta will be small, and it will be assumed that only the lowest possible combinations of l_K, l_π, and l_Λ need be considered. This approximation involves the additional assumption that those final state interactions which take place outside the region of the $\overline{\text{K}}\mathcal{N}$ interaction are negligible and do not introduce appreciable admixtures of higher orbital angular momenta. It will be convenient to employ the coupling scheme

$$\begin{cases} \boldsymbol{l}_K + \boldsymbol{s} = \boldsymbol{j}_K\,, & \boldsymbol{j}_K + \tfrac{1}{2} = \boldsymbol{J}\,, \\ \boldsymbol{l}_\pi + \boldsymbol{l}_\Lambda = \boldsymbol{L}_{\pi\Lambda}\,, & \boldsymbol{L}_{\pi\Lambda} + \tfrac{1}{2} = \boldsymbol{J}\,, \end{cases} \tag{9}$$

where s is the spin of the $\overline{\text{K}}$ meson. Table I lists for reactions (1), (2), and (3) the relevant values of l_K, j_K, J, l_π, l_Λ, and $L_{\pi\Lambda}$ for various combinations of $P_{K\Lambda}$ and s. It would appear that there exists some possibility of distinguishing these combinations by a study of the angular correlations of the reaction products and the energy dependence of the cross-sections.

TABLE I. – *The angular momenta which contribute appreciably at low energies to reactions* (1), (2) *and* (3) *for various combinations of the* K*-meson spin* s *and the* KΛ *relative parity* $P_{K\Lambda}$. *The combinations of* l_π, l_Λ, *and* $L_{\pi\Lambda}$ *contained in parentheses are not allowed in reaction* (1) *due to the identity of the pions.*

s	$P_{K\Lambda}$	l_K	j_K	J	l_π	l_Λ	$L_{\pi\Lambda}$
0	+1	0	0	$\frac{1}{2}$	0	0	0
0	−1	0	0	$\frac{1}{2}$	0 (1)	1 (0)	1 (1)
		1	1	$\frac{1}{2}$	0	0	0
2	+1	0	2	$\frac{3}{2}$	0 2 (1) (1)	2 0 (1) (1)	2 2 (2) (1)
		0	2	$\frac{5}{2}$	0 2 (1)	2 0 (1)	2 2 (2)
		1	1	$\frac{1}{2}$	0 (1)	1 (0)	1 (1)
		1	1	$\frac{3}{2}$	0 (1)	1 (0)	1 (1)
		1	2	$\frac{3}{2}$	0 (1)	1 (0)	1 (1)
		2	0	$\frac{1}{2}$	0	0	0
		2	1	$\frac{1}{2}$	0	0	0
2	−1	0	2	$\frac{3}{2}$	0 (1)	1 (0)	1 (1)
		1	1	$\frac{1}{2}$	0	0	0

3. – Theory.

3·1. *General treatment.* – Although the protons are unpolarized, the $\overline{K}$ mesons, having been produced in some reaction, must be regarded as being in an arbitrary state of polarization if they are assumed to have non-zero spin.

The treatment of this situation will be simplified by integrating the differential cross-section over the azimuthal angles $\varphi(\hat{k}_\pi)$ and $\varphi(\hat{k}_\Lambda)$, $\hat{k}_\pi$ and $\hat{k}_\Lambda$ being unit vectors in the directions of $\boldsymbol{p}_\pi$ and $\boldsymbol{p}_\Lambda$, respectively, and the axis of quantization being in the direction of $\boldsymbol{p}_K$. This integration eliminates any terms which may arise from interference between different initial spin states of the $\overline{K}$ meson and allows the incident beam to be treated as an incoherent mixture of spin states $\chi_{s,m}(\overline{K})$, each of which has a statistical weight I_m. It should be noted that because of the relation

$$P_L(\hat{k}_\pi \cdot \hat{k}_\Lambda) = [4\pi/(2L+1)] \sum_M Y^*_{LM}(\hat{k}_\pi) Y_{LM}(\hat{k}_\Lambda)$$

these azimuthal integrations preclude obtaining any simple correlation between $\hat{k}_\pi$ and $\hat{k}_\Lambda$, although such correlations do exist [1,2] and a more general treatment would yield explicit expressions for them. However, both $\hat{k}_\pi$ and $\hat{k}_\Lambda$ will be correlated with $\boldsymbol{p}_K$, and it is with these correlations that the following discussion will be concerned.

The total Hamiltonian H for the system will be written as

$$H = H_0 + H_1 .$$

The unperturbed Hamiltonian H_0 is a sum of terms, each term being the sum of the rest energy and the kinetic energy operator for some channel open to the system. The perturbing Hamiltonian H_1, which is actually defined by $H_1 \equiv H - H_0$, includes those operators which are responsible for the transitions being considered and also those which are responsible for final state interactions.

The initial wave function ψ_a which describes the unperturbed relative motion of the $\overline{K}p$ system in the spin state $\chi_{s,m}(\overline{K})\chi_{\frac{1}{2},m'}(p)$ is

$$\psi_a = \exp[ik_K z_K]\chi_{s,m}(\overline{K})\chi_{\frac{1}{2},m'}(p) , \qquad H_0\psi_a = E_a\psi_a . \tag{10}$$

An eigenfunction of H_0 which describes a state of unperturbed motion of the Λ^0 hyperon and two pions is

$$\psi_b = \exp[i\boldsymbol{k}_\pi \cdot \boldsymbol{r}_\pi] \exp[i\boldsymbol{k}_\Lambda \cdot \boldsymbol{r}_\Lambda]\chi_{\frac{1}{2},m'}(\Lambda) , \qquad H_0\psi_b = E_b\psi_b . \tag{11}$$

If $E_a = E_b$ the interaction H_1 may induce transitions from ψ_a to ψ_b, the differential cross-section for this process being given by

$$d\sigma_{ab} = (2\pi/\hbar)(1/v_K)|T_{ba}|^2\varrho , \tag{12}$$

where v_K is the relative velocity of the $\overline{K}p$ system, ϱ is the density of final states, and T_{ba} is defined by

$$T_{ba} = (\psi_b, H_1\Psi_a^{(+)}) , \tag{13}$$

$$\Psi_a^{(+)} = \psi_a + (E_a + i\varepsilon - H_0)^{-1}H_1\Psi_a^{(+)} . \tag{14}$$

The desired differential cross-section is found by averaging $\mathrm{d}\sigma_{ab}$ over the initial spin states, summing over the states of the Λ^0 spin, and, as mentioned previously, integrating over $\varphi(\hat{k}_\pi)$ and $\varphi(\hat{k}_\Lambda)$. It is given by

$$\mathrm{d}\sigma(\theta(\hat{k}_\pi),\,\theta(\hat{k}_\Lambda);\,p_\Lambda)=\tfrac{1}{2}\sum_{mm'm''}I_m\int_{\varphi(\hat{k}_\pi),\,\varphi(\hat{k}_\Lambda)}\mathrm{d}\sigma_{ab}\,. \tag{15}$$

This differential cross-section may then be integrated over all possible values of p_Λ and over either of the polar angles $\theta(\hat{k}_\pi)$ or $\theta(\hat{k}_\Lambda)$ to obtain

$$\mathrm{d}\sigma(\theta(\hat{k}_\pi))=\int_{p_\Lambda,\,\theta(\hat{k}_\Lambda)}\mathrm{d}\sigma(\theta(\hat{k}_\pi),\,\theta(\hat{k}_\Lambda);\,p_\Lambda)\,, \tag{16}$$

and

$$\mathrm{d}\sigma(\theta(\hat{k}_\Lambda))=\int_{p_\Lambda,\,\theta(\hat{k}_\pi)}\mathrm{d}\sigma(\theta(\hat{k}_\pi),\,\theta(\hat{k}_\Lambda);\,p_\Lambda)\,. \tag{17}$$

It will be convenient to expand T_{ba} in a series, each term of which depends on a particular combination of angular momenta. This is accomplished by first expanding eqs. (10) and (11) in spherical harmonics and following the coupling scheme (9) to obtain

$$\psi_a=\sum_{l_\mathrm{K}j_\mathrm{K}J}C_{l_\mathrm{K}s}(j_\mathrm{K},\,m;\,0,\,m)\,C_{j_\mathrm{K}\frac{1}{2}}(J,\,m+m';\,m,\,m')\,\varphi_{J,m+m'}(l_\mathrm{K},\,s,\,j_\mathrm{K};\,p_\mathrm{K})\,, \tag{18}$$

$$\begin{aligned}\varphi_{J,m+m'}(l_\mathrm{K},\,\mathrm{s},\,j_\mathrm{K};\,p_\mathrm{K})&=i^{l_\mathrm{K}}[4\pi(2l_\mathrm{K}+1)]^{\frac{1}{2}}j_{l_\mathrm{K}}(k_\mathrm{K}r_\mathrm{K})\cdot\\&\cdot\sum_{\mu\mu'}C_{l_\mathrm{K}s}(j_\mathrm{K},\,m+m'-\mu;\,m+m'-\mu-\mu',\,\mu')\cdot\\&\cdot C_{j_\mathrm{K}\frac{1}{2}}(J,\,m+m';\,m+m'-\mu,\,\mu)\,Y_{l_\mathrm{K},m+m'-\mu-\mu'}(\theta_\mathrm{K},\,\varphi_\mathrm{K})\,\chi_{s,\mu'}(\overline{\mathrm{K}})\,\chi_{\frac{1}{2},\mu}(\mathrm{p})\,,\end{aligned} \tag{19}$$

and

$$\begin{aligned}\psi_b=\sum_\alpha\,&C_{l_\pi l_\Lambda}(L_{\pi\Lambda},\,m_\pi+m_\Lambda;\,m_\pi,\,m_\Lambda)\cdot\\&\cdot C_{L_{\Lambda\pi}\frac{1}{2}}(J,\,m_\pi+m_\Lambda+m'';\,m_\pi+m_\Lambda,\,m'')\cdot\\&\cdot Y^*_{l_\pi,m_\pi}(\hat{k}_\pi)\,Y^*_{l_\Lambda,m_\Lambda}(\hat{k}_\Lambda)\,\varphi_{J,m_\pi+m_\Lambda+m''}(l_\pi,\,l_\Lambda,\,L_{\pi\Lambda};\,p_\pi,\,p_\Lambda)\,,\end{aligned} \tag{20}$$

$$\alpha=(l_\pi,\,l_\Lambda,\,L_{\pi\Lambda},\,J,\,m_\pi,\,m_\Lambda)\,,$$

$$\begin{aligned}\varphi_{J,m_\pi+m_\Lambda+m''}(l_\pi,\,l_\Lambda,\,L_{\pi\Lambda};\,p_\pi,\,p_\Lambda)&=(4\pi)^2\,i^{l_\pi+l_\Lambda}j_{l_\pi}(k_\pi r_\pi)\,j_{l_\Lambda}(k_\Lambda r_\Lambda)\cdot\\&\cdot\sum_{\nu\nu'}C_{l_\pi l_\Lambda}(L_{\pi\Lambda},\,m_\pi+m_\Lambda+m''-\nu';\,m_\pi+m_\Lambda+m''-\nu-\nu',\,\nu)\cdot\\&\cdot C_{L_{\pi\Lambda}\frac{1}{2}}(J,\,m_\pi+m_\Lambda+m'';\,m_\pi+m_\Lambda+m''-\nu',\,\nu')\cdot\\&\cdot Y_{l_\pi,m_\pi+m_\Lambda+m''-\nu-\nu'}(\theta_\pi,\,\varphi_\pi)\,Y_{l_\Lambda,\nu}(\theta_\Lambda,\,\varphi_\Lambda)\,\chi_{\frac{1}{2},\nu'}(\Lambda)\,.\end{aligned} \tag{21}$$

In eqs. (18), (19), (20), and (21) the quantities of the form $C_{j_1 j_2}(j_3,\ m_1+m_2;\ m_1,\ m_2)$ are vector addition coefficients, the notation being that of BLATT and WEISSKOPF [10]. Since the perturbing interaction H_1 transforms under spatial rotations as a scalar, it is a simple matter to combine eqs. (13), (14), (18), and (20) and write T_{ba} as

$$
\begin{aligned}
(22)\qquad T_{ba} = \sum_{\beta m_\Lambda} & C_{l_\pi l_\Lambda}(L_{\pi\Lambda},\ m+m'-m'';\ m+m'-m''-m_\Lambda,\ m_\Lambda)\cdot \\
& \cdot C_{L_{\pi\Lambda}\,\frac{1}{2}}(J,\ m+m';\ m+m'-m'',\ m'')\, C_{l_K\, s}(j_K,\ m;\ 0,\ m)\cdot \\
& \cdot C_{j_K\,\frac{1}{2}}(J,\ m+m';\ m,\ m')\, Y_{l_\pi,\, m+m'-m''-m_\Lambda}(\hat{k}_\pi)\, Y_{l_\Lambda, m_\Lambda}(\hat{k}_\Lambda)\, t(\beta;\ p_K,\ p_\pi,\ p_\Lambda)\,, \\
& \beta = (l_K,\ j_K,\ J,\ l_\pi,\ l_\Lambda,\ L_{\pi\Lambda})\,,
\end{aligned}
$$

where

$$
(23)\qquad t(\beta;\ p_K,\ p_\pi,\ p_\Lambda) = (\varphi_{J,M}(l_\pi,\ l_\Lambda,\ L_{\pi\Lambda};\ p_\pi,\ p_\Lambda),\ H_1 \Phi^{(+)}_{J,M}(l_K,\ s,\ j_K;\ p_K))\,,
$$

$$
\begin{aligned}
(24)\qquad \Phi^{(+)}_{J,M}(l_K,\ s,\ j_K;\ p_K) &= \varphi_{J,M}(l_K,\ s,\ j_K;\ p_K) + \\
&+ (E_a + i\varepsilon - H_0)^{-1} H_1 \Phi^{(+)}_{J,M}(l_K,\ s,\ j_K;\ p_K)\,.
\end{aligned}
$$

The matrix element $t(\beta;\ p_K,\ p_\pi,\ p_\Lambda)$ is independent of magnetic quantum numbers.

The formal expression for $\mathrm{d}\sigma(\theta(\hat{k}_\pi),\ \theta(\hat{k}_\Lambda);\ p_\Lambda)$ in terms of the $t(\beta;\ p_K,\ p_\pi,\ p_\Lambda)$ is found by combining eqs. (12), (15), and (22) and employing well known techniques to perform the sums over all magnetic quantum numbers except m. However, this expression is exceedingly complicated and will not be given here. The somewhat simpler expression for $\mathrm{d}\sigma(\theta(\hat{k}_\Lambda))$ is

$$
\begin{aligned}
(25)\qquad \mathrm{d}\sigma(\theta(\hat{k}_\Lambda)) = [\pi/(2\hbar v_K)] & \sum_{\gamma\gamma' \mathscr{L}_\Lambda m} I_m \int \varrho\cdot \\
& \cdot (-1)^{l_\pi + \mathscr{L}_\Lambda + m}(2J+1)(2J'+1)(2\mathscr{L}_\Lambda+1)\cdot \\
& \cdot [(2l_\Lambda+1)(2l'_\Lambda+1)(2L_{\pi\Lambda}+1)(2L'_{\pi\Lambda}+1)(2j_K+1)(2j'_K+1)]^{\frac{1}{2}}\cdot \\
& \cdot \begin{Bmatrix} L_{\pi\Lambda} & L'_{\pi\Lambda} & \mathscr{L}_\Lambda \\ l'_\Lambda & l_\Lambda & l_\pi \end{Bmatrix} \begin{Bmatrix} J & J' & \mathscr{L}_\Lambda \\ L'_{\pi\Lambda} & L_{\pi\Lambda} & \frac{1}{2} \end{Bmatrix} \begin{Bmatrix} J & J' & \mathscr{L}_\Lambda \\ j'_K & j_K & \frac{1}{2} \end{Bmatrix}\cdot \\
& \cdot \begin{pmatrix} s & l_K & j_K \\ m & 0 & -m \end{pmatrix} \begin{pmatrix} s & l'_K & j'_K \\ m & 0 & -m \end{pmatrix} \begin{pmatrix} j_K & j'_K & \mathscr{L}_\Lambda \\ -m & m & 0 \end{pmatrix} \begin{pmatrix} l'_\Lambda & l_\Lambda & \mathscr{L}_\Lambda \\ 0 & 0 & 0 \end{pmatrix}\cdot \\
& \cdot t(\gamma;\ p_K,\ p_\pi,\ p_\Lambda)\, t^*(\gamma';\ p_K,\ p_\pi,\ p_\Lambda) P_{\mathscr{L}_\Lambda}(\hat{k}_\Lambda \cdot \hat{k}_K)\,, \\
& \gamma = (l_K,\ j_K,\ J,\ l_\pi,\ l_\Lambda,\ L_{\pi\Lambda})\,, \\
& \gamma' = (l'_K,\ j'_K,\ J',\ l_\pi,\ l'_\Lambda,\ L'_{\pi\Lambda})\,.
\end{aligned}
$$

[10] J. M. BLATT and V. F. WEISSKOPF: *Theoretical Nuclear Physics* (New York, 1952), Appendix A.

In eq. (25) the quantities contained in parentheses and curly brackets are, respectively, the familiar $3j$ and $6j$ symbols, the notation being that of EDMONDS ([11]).

The formal expression for $\mathrm{d}\sigma(\theta(\hat{k}_\pi))$ is quite similar to that for $\mathrm{d}\sigma(\theta(\hat{k}_\Lambda))$. It may be obtained by subjecting eq. (25) to the transformation

$$l_\Lambda \to l_\pi\,, \qquad l'_\Lambda \to l'_\pi\,, \qquad l_\pi \to l_\Lambda\,,$$

$$\mathscr{L}_\Lambda \to \mathscr{L}_\pi\,, \qquad \hat{k}_\Lambda \to \hat{k}_\pi\,,$$

$$\gamma' = (l'_\mathrm{K},\, j'_\mathrm{K},\, J',\, l_\pi,\, l'_\Lambda,\, L'_{\pi\Lambda}) \to \gamma' = (l'_\mathrm{K},\, j'_\mathrm{K},\, J',\, l'_\pi,\, l'_\Lambda,\, L'_{\pi\Lambda})\,,$$

and multiplying each term of the expansion by the factor $(-1)^{L_{\pi\Lambda}+L'_{\pi\Lambda}+l_\pi+l'_\pi}$. The similarity of the two expressions is due to the symmetrical fashion in which $\boldsymbol{l}_\pi$ and $\boldsymbol{l}_\Lambda$ were introduced in the coupling scheme (9). The change in phase of the individual terms of the expansion may be traced to the change in phase of the vector addition coefficient $C_{l_\pi l_\Lambda}(L_{\pi\Lambda},\, m_\pi+m_\Lambda;\, m_\pi,\, m_\Lambda)$ in eq. (20) when l_π and l_Λ are interchanged.

3·2. *Approximations.* – The low values of the wave numbers k_K, k_π, and k_Λ, and the small range of the $\overline{\mathrm{K}}\mathscr{N}$ interaction suggest the possibility that for low $\overline{\mathrm{K}}$ meson energies the momentum dependence of the matrix element $t(\beta;\, p_\mathrm{K},\, p_\pi,\, p_\Lambda)$ is determined almost entirely by the penetrability of the centrifugal barrier. Since the spherical Bessel function $j_L(KR)$ is approximated closely at small distances by

$$j_L(KR) \simeq (KR)^L/(2L+1)!!$$

it is seen from eqs. (19), (21), and (23) that the simplest assumption possible is

$$t(\beta;\, p_\mathrm{K},\, p_\pi,\, p_\Lambda) \propto (p_\mathrm{K})^{l_\mathrm{K}}(p_\pi)^{l_\pi}(p_\Lambda)^{l_\Lambda}\,. \tag{26}$$

Of course, p_K, p_π, and p_Λ are not all independent, being related non-relativistically by eqs. (4) and (6).

The non-relativistic expression for the phase space density is, apart from numerical factors, given by

$$\varrho \propto p_\Lambda^2[2\mu_\Lambda W - p_\Lambda^2]^{\frac{1}{2}}\,\mathrm{d}p_\Lambda\,\mathrm{d}\Omega(\hat{k}_\pi)\,\mathrm{d}\Omega(\hat{k}_\Lambda)\,. \tag{27}$$

These approximations, combined with the restrictions on possible values

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([11]) A. R. EDMONDS: *Angular Momentum in Quantum Mechanics* (Princeton, 1957), Chs. 3 and 6.

of l_K, l_π, and l_Λ given in Table I, allow the expression (25) to be integrated and given simply in terms of the energy of the incident $\overline{K}$ meson.

For the values of s and $P_{K\Lambda}$ being considered, the form of

$$d\sigma(\theta(\hat{k}_\Lambda))/d\cos\theta(\hat{k}_\Lambda)$$

in reactions (1), (2), and (3) may be shown to be

$$d\sigma(\theta(\hat{k}_\Lambda))/d\cos\theta(\hat{k}_\Lambda) \propto [(1+\mathscr{E}_K)^2/(\mathscr{E}_K)^{\frac{1}{2}}] \sum A_{\mathscr{L}_\Lambda}(\mathscr{E}_K) P_{\mathscr{L}_\Lambda}(\hat{k}_\Lambda \cdot \hat{k}_K) , \tag{28}$$

where $\mathscr{E}_K$ is defined by

$$\mathscr{E}_K \equiv [p_K^2/(2\mu_K)]/Q . \tag{29}$$

The coefficients $A_{\mathscr{L}_\Lambda}(\mathscr{E}_K)$ are listed in Table II. In Table II those coefficients a_{01}, b_{01}, etc., which are associated with $A_0(\mathscr{E}_K)$ are all positive numbers, while the other coefficients b_{11}, c_{11}, c_{12}, etc., may be positive, negative, or, due

TABLE II. – *The energy dependence of the coefficients* $A_{\mathscr{L}_\Lambda}(\mathscr{E}_K)$ *which appear in eq.* (28).

s	$P_{K\Lambda}$	$A_0(\mathscr{E}_K)$	$A_1(\mathscr{E}_K)$	$A_2(\mathscr{E}_K)$	$A_3(\mathscr{E}_K)$	$A_4(\mathscr{E}_K)$
0	+1	a_{01}	0	0	0	0
0	−1	$b_{01}\mathscr{E}_K + b_{02}(1+\mathscr{E}_K)$	$b_{11}[\mathscr{E}_K(1+\mathscr{E}_K)]^{\frac{1}{2}}$	0	0	0
2	+1	$c_{01}\mathscr{E}_K^2 + c_{02}\mathscr{E}_K(1+\mathscr{E}_K) + c_{03}(1+\mathscr{E}_K)^2$	$[c_{11}\mathscr{E}_K + c_{12}(1+\mathscr{E}_K)] \cdot [\mathscr{E}_K(1+\mathscr{E}_K)]^{\frac{1}{2}}$	$[c_{21}\mathscr{E}_K + c_{22}(1+\mathscr{E}_K)] \cdot (1+\mathscr{E}_K)$	$c_{31}\mathscr{E}_K^{\frac{1}{2}}(1+\mathscr{E}_K)^{\frac{3}{2}}$	$c_{41}(1+\mathscr{E}_K)^2$
2	−1	$d_{01}\mathscr{E}_K + d_{02}(1+\mathscr{E}_K)$	$d_{11}[\mathscr{E}_K(1+\mathscr{E}_K)]^{\frac{1}{2}}$	$d_{21}(1+\mathscr{E}_K)$	0	0

to cancellation of terms, zero. The numerical values of these coefficients will differ in reactions (1), (2), and (3) because the initial isotopic spin state in (1) and (2) is not the same as in (3), and also because the identity of the pions in (1) introduces restrictions on the possible values of l_π and l_Λ. Of course, in any one of these reactions these coefficients are not all independent; but in the formalism employed here the connection between them is usually not a simple one. The energy dependence of the total cross-section is given by the first term of the expansion in eq. (28).

The form of the corresponding approximate expression for $d\sigma(\theta(\hat{k}_\pi))/d\cos\theta(\hat{k}_\pi)$ is almost the same as that for $d\sigma(\theta(\hat{k}_\Lambda))/d\cos\theta(\hat{k}_\Lambda)$. The only difference is in reaction (1) where the identity of the not readily detectable neutral pions

causes the coefficients

$$A_1(\mathscr{E}_\mathrm{K}),\ \begin{cases} s=0\,, & P_{\mathrm{K}\Lambda}=-1\,,\\ s=2\,, & P_{\mathrm{K}\Lambda}=\pm 1\,,\end{cases}$$

$$A_2(\mathscr{E}_\mathrm{K}),\quad s=2\,,\quad P_{\mathrm{K}\Lambda}=-1\,,$$

$$A_3(\mathscr{E}_\mathrm{K}),\quad s=2\,,\quad P_{\mathrm{K}\Lambda}=+1\,,$$

to vanish in $\mathrm{d}\sigma(\theta(k_\pi))/\mathrm{d}\cos\theta(k_\pi)$. Otherwise, the preceding discussion applies entirely.

4. – Discussion.

The assumptions and approximations made in Sections 2 and 3 are merely a simple extension of those originally introduced by Okun' and Pomerančuk [1] and are identical with those employed by Fonda and Russell [2]. If they are valid, the four combinations of s and $P_{\mathrm{K}\Lambda}$ that were considered can, at least in principle, all be distinguished from one another. If $P_{\mathrm{K}\Lambda}=+1$ the differential cross-sections and also the total cross-section for $s=0$ all differ from those for $s=2$. However, if $P_{\mathrm{K}\Lambda}=-1$ spin 2 is distinguished from spin zero only by the presence of a second order Legendre polynomial in the angular distributions, and, as mentioned previously, the coefficient of this term may well be small. In any case it is seen that by measuring the energy dependence of the total cross-section a determination of $P_{\mathrm{K}\Lambda}$ may be made which is independent of whether the K meson spin is zero or 2.

It is hoped that the relations given in Section 3 will be approximately correct for $\overline{\mathrm{K}}$ mesons with laboratory energies below, say, 30 MeV. In this energy range the relativistic corrections to the phase space density will amount to only a few percent, as in the very similar case of τ decay [12,13].

Since pion-pion final state interactions do not appear to play an important role in τ decay [14], it would seem reasonable to expect that the same would hold true here, the range of pion energies being almost identical. On the other hand, since little is known with certainty regarding the low energy $\pi\Lambda$ interaction, the assumption that its effects may be neglected is considerably more daring, and additional information would be desirable before an entirely satis-

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[12] R. H. Dalitz: *Reports on Progress in Physics*, vol. **20** (London, 1957), p. 163.

[13] E. Fabri: *Nuovo Cimento*, **11**, 479 (1954).

[14] See, for example, S. McKenna, S. Natali, M. O'Connell, J. Tietge and N. C. Varshneya: *Nuovo Cimento*, **10**, 763 (1958), and references contained therein.

factory analysis of these reactions, either in flight or at rest, could be made.

Apart from elementary considerations based on the penetrability of the centrifugal barrier, rather little *a priori* justification can be given for (26), the principal approximation employed in this paper. However, these same considerations have been applied successfully in the past to other phenomena, notably τ decay ([12]) and certain low energy π meson reactions ([15]), and it would appear that a study of (1), (2), and (3), if feasible within a reasonable length of time, would serve as a tentative determination of $P_{K\Lambda}$ (and perhaps s), subject, of course, to corroboration by other, independent methods.

* * *

Grateful acknowledgement is made to Dr. M. H. Ross and Dr. G. L. Shaw for helpful discussions.

([15]) See, for example, M. Gell-Mann and K. M. Watson: *Annual Review of Nuclear Science*, vol. **4** (Stanford, 1954), p. 219.

RIASSUNTO (*)

Una recente proposta di determinare $P_{K\Lambda}$, la parità relativa KΛ, mediante lo studio della reazione $K+p \to \Lambda^0 + 2\pi$, in volo ed a basse energie, viene generalizzata per includere la pur remota possibilità di un mesone K di spin 2. Si trovano le formule appropriate delle sezioni trasversali differenziali, con un trattamento non relativistico e rigidamente fenomenologico. L'approssimazione degli elementi della matrice è basata su considerazioni sulla penetrazione della barriera. Si trascurano le interazioni dello stato finale. Se i presupposti, che son uguali a quelli assunti precedentemente, sono validi, $P_{K\Lambda}$ può essere determinata indipendentemente dal fatto che lo spin dei mesoni K sia zero o 2. In aggiunta si ha la possibilità di distinguere lo spin zero dallo spin 2.

(*) *Traduzione a cura della Redazione.*

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Observations concernant la radiation γ de type $E2$ des noyaux atomiques déformés (Modèle de Nilsson).

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Résumé. — On a calculé les probabilités de transitions γ de type $E2$ en faisant usage de la représentation de Nilsson qui tient compte du couplage entre les états à differents N. Pour cela on a utilisé les résultats obtenus au calcul des moments électriques quadrupolaires. L'étude concrète de certaines transitions γ de type $E2$ de quelques noyaux impairs de la région $150 < A < 188$ montre qu'il est nécéssaire de faire usage de la représentation mentionnée ci-dessus.

1. – Considérations théoriques.

L'opérateur électrique quadrupolaire au moyen duquel on calcule, la probabilité réduite de transition $\mathscr{B}(E2)$, dans le système de référence fixe dans l'espace, K'', s'écrit de la manière suivante

$$\hat{\mathscr{M}}(E2) = \sqrt{\frac{5}{16\pi}}\, e\hat{Q}_M = e\sum_{i=1}^{Z} r_i''^2\, Y_{2,M}(\vartheta_i'', \varphi_i'')\,. \tag{1}$$

Dans le système nucléaire mobile, K', cet opérateur, en raison des propriétés de transformation de rotation des tenseurs irreductibles de deuxième rang, devient

$$\hat{\mathscr{M}}(E2) = e\sum_{i=1}^{Z} r_i'^2 \sum_{\nu=-2}^{2} Y_{2,\nu}(\vartheta_i', \varphi_i')\mathscr{D}^2_{M,\nu}\,. \tag{2}$$

Le modèle de Nilsson [1] comprend deux représentations: l'une, qui est ap-

[1] S. G. Nilsson: *Det. Kong. Dan. Mat. Fys. Medd.*, **29**, 16 (1955).

proximative, néglige le couplage entre les états à différents N, et fait usage, dans le système mobile, du changement de variable suivant

$$r^2 = \frac{M\omega_0(\delta)}{\hbar} r'^2 ; \qquad \omega_0(\delta) \approx \overset{\circ}{\omega}_0 \left(1 + \frac{2}{9}\delta^2\right) . \tag{3}$$

L'autre est une représentation corrigée qui tient compte du couplage entre les états à différents N, par l'introduction de ε à la place du paramètre de déformation δ:

$$\varepsilon = \delta + \tfrac{1}{6}\delta^2 + \mathcal{O}(\delta^3)$$

et par le changement de variable

$$\xi = x' \sqrt{\frac{M\omega_x}{\hbar}} ; \qquad \eta = y' \sqrt{\frac{M\omega_y}{\hbar}} ; \qquad \zeta = z' \sqrt{\frac{M\omega_z}{\hbar}} , \tag{4}$$

où

$$\omega_x = \omega_y = \omega_0(\varepsilon)(1 + \tfrac{1}{3}\varepsilon) ; \qquad \omega_z = \omega_0(\varepsilon)(1 - \tfrac{2}{3}\varepsilon) ; \qquad \omega_0(\varepsilon) \approx \overset{\circ}{\omega}_0(1 + \tfrac{1}{9}\varepsilon^2) .$$

Le changement de variable (3), en ne modifiant que la grandeur du rayon vecteur, ne modifie pas la partie angulaire des différentes grandeurs exprimées en coordonnées sphériques et l'on obtient pour l'opérateur $\hat{\mathcal{M}}(E2)$ l'expression suivante:

$$\hat{\mathcal{M}}(E2) = \frac{\hbar}{M\omega_0(\delta)} e \sum_{i=1}^{Z} r_i^2 \sum_{\nu=-2}^{2} Y_{2,\nu}(\vartheta_i', \varphi_i') \mathcal{D}_{M,\nu}^2 . \tag{5}$$

Dans la représentation corrigée, en raison de l'amplification par des facteurs différents des coordonnées x', y', z', dû au changement de variable, la partie angulaire est aussi modifiée et, si l'on retient seulement les termes de premier ordre en ε, $\hat{\mathcal{M}}(E2)$ devient:

$$\begin{aligned} \mathcal{M}(E2) = \frac{\hbar}{M\omega_0(\varepsilon)} e \sum_{i=1}^{Z} \Bigg\{ \left[\varrho_i^2 U_{2,0} + \frac{1}{3}\varepsilon \left(\varrho_i^2 U_{2,0} + \sqrt{\frac{5}{4\pi}} \varrho_i^2 \right) \right] \mathcal{D}_{M,0}^2 + \\ + \left(1 + \frac{1}{6}\varepsilon\right) (\varrho_i^2 U_{2,1} \mathcal{D}_{M,1}^2 + \varrho_i^2 U_{2,-1} \mathcal{D}_{M,-1}^2) + \\ + \left(1 - \frac{1}{3}\varepsilon\right) (\varrho_i^2 U_{2,2} \mathcal{D}_{M,2}^2 + \varrho_i^2 U_{2,-2} \mathcal{D}_{M,-2}) \Bigg\} . \end{aligned} \tag{6}$$

Ici $U_{2,M}$ est la fonction sphérique normée, d'ordre $(2, M)$, exprimée à l'aide des angles du système des coordonnées ξ, η, ζ.

On considère la fonction d'onde du noyau, à l'aide de laquelle on a calculé

les probabilités de transition, comme ayant la forme

$$|\Omega; JKM\rangle = \sqrt{\frac{2J+1}{16\pi^2}}\{\chi_\Omega \mathscr{D}^J_{M,K} + (-1)^{J-\sum_i j_i}\chi_{-\Omega}\mathscr{D}^J_{M,-K}\}. \tag{7}$$

En ce qui concerne la partie individuelle de la fonction d'onde, χ_Ω, on admet, comme dans [2]) qu'elle consiste d'un produit de deux determinants de Slater construits à l'aide des fonctions d'onde uniparticule données par le modèle de Nilsson. Celles-ci ont la forme de combinaisons linéaires des vecteurs propres $|N l \Lambda \Sigma\rangle$ du hamiltonien correspondant au mouvement dans un potentiel oscillateur tridimensionnel à symétrie sphérique. Dans la représentation corrigée les fonctions d'onde ont la même forme, les vecteurs propres de l'hamiltonien symétrique sphérique étant en ce cas $|N_t l_t \Lambda_t \Sigma\rangle$ où l_t et Λ_t sont les nombres quantiques correspondant aux opérateurs $\hat{l}_t^2$ et $(\hat{l}_t)_z$ définis comme:

$$\hat{l}_t^2 = (\hat{l}_t)_x^2 + (\hat{l}_t)_y^2 + (\hat{l}_t)_z^2; \qquad (\hat{l}_t)_x = -i\left(\eta\frac{\partial}{\partial\zeta} - \zeta\frac{\partial}{\partial\eta}\right).$$

On interprète les coéfficients $a_{l\Lambda}$ des combinaisons linéaires dans la représentation $|N l \Lambda \Sigma\rangle$ comme étant les coefficients $a_{l_t\Lambda_t}$ dans la représentation $|N_t l_t \Lambda_t \Sigma\rangle$.

Pour le calcul de $\mathscr{B}(E2)$ on doit faire la distinction entre deux groupes de transitions: dans le premier groupe on prend en considération celles qui ont le même χ_Ω tant dans l'état initial (i) que dans l'état final (f), c'est-à-dire les transitions qui se produisent entre les niveaux d'une même bande rotationnelle. En tenant compte que $\Omega = K = \Lambda + \Sigma$ on obtient dans la première représentation:

$$\mathscr{B}(E2) = \sum_{M,M_f} |\langle K; J_f K M_f | \hat{\mathscr{M}}(E2) | K; J_i K M_i\rangle|^2 \tag{8}$$

où

$$\langle K; J_f K M_f | \hat{\mathscr{M}}(E2) | K; J_i K M_i\rangle = \langle K; J_f K M_f | \sum_{i=1}^{Z} r_i^2 \sum_\nu Y_{2,\nu}\mathscr{D}^2_{M,\nu} | K; J_i K M_i\rangle =$$

$$= \sqrt{\frac{2J_i+1}{2J_f+1}}\langle J_i 2 M_i M | J_f M_f\rangle\langle J_i 2 K 0 | J_f K\rangle\langle\chi_K | \sum_{i=1}^{Z} r_i^2 Y_{2,0} | \chi_K\rangle.$$

On obtient cette expression-ci à partir de la formule bien connue

$$\int \mathscr{D}^{J'*}_{M',K'}\mathscr{D}^\lambda_{\mu,\nu}\mathscr{D}^J_{M,K}\,\mathrm{d}\Omega^3 = \frac{8\pi^2}{2J'+1}\langle J\lambda M\mu | J'M'\rangle\langle J\lambda K\nu | J'K'\rangle. \tag{9}$$

([2]) D. Bogdan: *Nuovo Cimento*, **10**, 985 (1958).

Si l'on tient compte du fait que l'opérateur quadrupole à la forme d'une fonction symétrique des coordonnées des Z protons, on peut aisément montrer que

$$\mathscr{B}(E2) = \frac{5}{16\pi} e^2 \left[2 \frac{\hbar}{M\omega_0(\delta)} \sum_{i=1}^{Z} \mathscr{A}_i(\delta)^2 \right]^2 \langle J_i 2K0 | J_f K\rangle^2 , \tag{10}$$

où $\mathscr{A}_i(\delta)$ est donné par

$$\mathscr{A}_i(\delta) = \sum_{l_i l_i'} \langle l_i 2\,0\,0 | l_i' 0\rangle \sqrt{\frac{2l_i+1}{2l_i'+1}} \langle N_i l_i' | r_i^2 | N_i l_i\rangle \sum_{\Lambda_i \Lambda_i' \Sigma_i \Sigma_i'} a'_{l_i' \Lambda_i' \Sigma_i'} a_{l_i \Lambda_i \Sigma_i} \langle l_i 2 \Lambda_i 0 | l_i' \Lambda_i'\rangle \delta_{\Sigma'\Sigma_i'} .$$

En calculant $\mathscr{B}(E2)$ à l'aide de (6) la formule (9) ne maintient que le terme qui contient $\mathscr{D}^2_{M,0}$ comme facteur, et pour les mêmes raisons que plus haut l'on obtient

$$\mathscr{B}(E2) = \tag{11}$$
$$= \frac{5}{16\pi} e^2 \left\{ 2 \frac{\hbar}{M\omega_0(\varepsilon)} \sum_{i=1}^{Z} \left[\mathscr{A}_i(\varepsilon) + \frac{1}{3}\varepsilon \left(\mathscr{A}_i(\varepsilon) + \langle N_t l_t | \varrho_i^2 | N_t l_t\rangle \right) \right] \right\}^2 \langle J_i 2K0 | J_f K\rangle^2 .$$

Dans [2] j'ai comparé les valeurs expérimentales des quadrupoles intrinsèque, Q_0, déduits à partir des valeurs expérimentales des grandeurs $\mathscr{B}(E2)$, aux valeurs théoriques des mêmes grandeurs qui, selon (10) et (11), ont l'aspect suivant

$$Q_0(\delta) = 2 \frac{\hbar}{M\omega_0(\delta)} \sum_{i=1}^{Z} \mathscr{A}_i(\delta) , \tag{12a}$$

$$Q_0(\varepsilon) = 2 \frac{\hbar}{M\omega_0(\varepsilon)} \sum_{i=1}^{Z} \left[\mathscr{A}_i(\varepsilon) + \frac{\varepsilon}{3} \left(\langle N_l l_t | \varrho_i^2 | N_t l_t\rangle + \mathscr{A}_i(\varepsilon) \right) \right] . \tag{12b}$$

Ici $\mathscr{A}_i(\varepsilon)$ a une forme identique à $\mathscr{A}_i(\delta)$, avec la seule différence que dans la première fonction les nombres quantiques sont l_t, N_t, Λ_t et que l'on a ϱ_i^2 au lieu de r_i^2. La comparaison avec l'expérience a montré que les valeurs de la distance entre les niveaux de l'oscillateur harmonique isotrope, $\hbar\overset{0}{\omega}_0$, obtenues à l'aide de l'expression théorique (12 *b*) sont sensiblement différentes de celles obtenues à l'aide de l'expression théorique (12 *a*) (*).

(*) Les expressions (9) et (11) de [2] diffèrent des expressions (12*a*) et (12*b*) pour les raisons suivantes. En (9) et (11) de [2] on a noté par Q_0 la grandeur $\langle \hat{\mathscr{M}}(E2)\rangle$ (lorsque j'ai calculé les valeurs $\hbar\overset{0}{\omega}_0$ j'ai cependant tenu compte du fait que les valeurs expérimentales, $Q_{0\,\mathrm{exp.}}$ doivent être comparées aux grandeurs $(16\pi/5)^{\frac{1}{2}}\langle \hat{\mathscr{M}}(E2)\rangle$). En outre, pour exprimer $Q_0(\varepsilon)$, on a fait usage, dans le présent article, du développement en série du facteur $1/(1-\frac{1}{3}\varepsilon-\frac{2}{9}\varepsilon^2)$, interrompu au terme de premier ordre en ε. De même, la formule (11) de [2] contient une erreur: la multiplication par 2 du terme

Dans le second groupe de transitions on prend en considération celles pour lesquelles χ_Ω n'est pas le même dans l'état initial que dans l'état final. Dans la représentation approximative, la formule (9) ne maintient de la sommation par rapport a ν de (5) qu'un des termes $\nu = K_f - K_i = \pm 1$ ou ± 2. De même si l'on tient compte du fait que l'opérateur électrique quadrupolaire a la forme d'une fonction symétrique des coordonnés des Z protons, il s'ensuit, selon un théorème bien connu (voir [3], Chap. VI) que l'élément de matrice de cet operateur:

$$\langle \chi_{\Omega_f} | \sum_{i=1}^{Z} r_i^2 Y_{2.K_f - K_i} | \chi_{\Omega_i} \rangle = 0$$

si au cours de la transition plus d'un proton change l'état quantique. On obtient alors, à partir de

$$\mathscr{B}(E2) = \sum_{M.M_f} | \langle K_f; J_f K_f M_f | \hat{\mathscr{M}}(E2) | K_i; J_i K_i M_i \rangle |^2 \tag{13}$$

la formule pour les transitions uniparticules de type $E2$, dont l'expression est écrite par NILSSON dans le cas $2 < K_i + K_f$, comme

$$\mathscr{B}(E2) = e^2 \left(1 + \frac{Z}{A^2}\right)^2 \left(\frac{\hbar}{M\omega_0(\delta)}\right)^2 \frac{5}{4\pi} \langle J_i 2 K_i K_f - K_i | J_f K_f \rangle^2 G_{E2}^2 , \tag{14}$$

où

$$G_{E2} = \sum_{l_f, l_i} \langle N_f l_f | r | N_i l_i \rangle \sqrt{\frac{2l_i + 1}{2l_f + 1}} \langle l_i 200 | l_f 0 \rangle \sum_{\Lambda_f, \Lambda_i, \Sigma_f, \Sigma_i} a_{l_f \Lambda_f} a_{l_i \Lambda_i} \langle l_i 2_i \Lambda_i K_f - K_i | l_f \Lambda_f \rangle .$$

contenu dans la parenthèse, de premier ordre en ε. Si l'on tient compte de ce dernier fait, les valeurs $\hbar\overset{0}{\omega}_0(Q_0(\varepsilon))$ sont différentes de celles calculées dans [2]. Le tableau suivant contient les valeurs $\hbar\overset{0}{\omega}_0(Q_0(\varepsilon_e))$ modifiées, aussi bien que les valeurs $\hbar\overset{0}{\omega}_0(Q_0(\delta_e))$ et $\hbar\overset{0}{\omega}_0 = 41 A^{-\frac{1}{3}}$ (MeV) données à titre de comparaison:

Noyau	$^{153}_{63}$Eu	$^{159}_{65}$Tb	$^{165}_{67}$Ho	$^{175}_{71}$Lu	$^{179}_{72}$Hf	$^{177}_{72}$Hf	$^{169}_{69}$Tm
ε_e	0.33	0.34	0.30	0.26	0.24	0.25	0.30
$\hbar\overset{0}{\omega}_0(Q_0(\varepsilon_e))$ (MeV)	9.26	8.78	7.01	6.83	6.45	7.09	8.46
$\hbar\overset{0}{\omega}_0(Q_0(\delta_e))$ (MeV)	4.70	4.33	3.96	3.45	3.27	3.59	4.22
$\hbar\overset{0}{\omega}_0 = 41 A^{-\frac{1}{3}}$ (MeV)	7.67	7.57	7.48	7.33	7.28	7.30	7.42

[3] U. E. CONDON and G. H. SHORTLEY: *The Theory of Atomic Spectra* (Cambridge, 1953).

Dans la représentation corrigé le calcul de la probabilité réduite se fait aussi a partire de (13), la différence consistant dans l'opérateur utilisé ((16) au lieu de (5)), et dans la signification des fonctions d'onde individuelles. Si l'on néglige le terme proportionnel à ε^2 on peut écrire le résultat de la manière suivante

$$\mathscr{B}(E2) = e^2\left(1+\frac{Z}{A^2}\right)^2\left(\frac{\hbar}{M\omega_0(\varepsilon)}\right)^2\frac{5}{4\pi}\langle J_i 2K_i K_f - K_i | J_f K_f\rangle^2 u(\varepsilon) G_{E2}^2 \,. \tag{15}$$

Ici G_{E2} a la même forme que dans (14), ou N, l, Λ, r, sont remplacés par N_t, l_t, Λ_t, ϱ, et $u(\varepsilon)$ est

$$u(\varepsilon) = \begin{cases} 1+\frac{1}{3}\varepsilon & \text{si} \quad K_f - K_i = \pm 1 \\ 1-\frac{2}{3}\varepsilon & \text{si} \quad K_f - K_i = \pm 2 \,. \end{cases}$$

De même il paraît naturel d'amettre que $\overset{0}{\omega}_0$ qui intervient dans (15) a une valeur qui coïncide avec celle déterminée de $Q_{0\,\mathrm{exp}}$ et de l'expression théorique $Q_0(\varepsilon_e)$, tandis que la valeur de $\overset{0}{\omega}_0$ dans (14) s'obtient de $Q_{0\,\mathrm{exp}}$ et de l'expression théorique $Q_0(\delta_e)$. Pour simplifier les calculs il a été nécéssaire aussi d'admettre que le paramètre de déformation a la même valeur dans l'état initial que dans l'état final ce qui n'est pas rigoureusement exact. Je me suis permis d'admettre cette approximation car je n'ai pris en considération que les paires d'états dont la différence d'énergie entre les niveaux est petite (tout au plus quelques centaines de keV).

2. – L'étude de quelques transitions concrètes.

Les résultats précédents ont été appliqués au calcul des probabilités de quelques transitions γ de type $E2$ du second groupe (χ_Ω différent dans l'état initial et l'état final) de certains noyaux atomiques impairs de la région $150 < A < 188$. J'ai pris en considération seulement les transitions dont les paires de niveaux ont les paramètres caractéristiques (spin, parité, $\tau = \ln 2/T$ (T = la probabilité de transition)) déterminés aussi complètement que possible et qui se produisent chez les noyaux dont le paramètre de déformation n'est pas sensiblement éloigné de l'intervalle $4 \leqslant \eta \leqslant 6$. A cette fin j'ai fait usage des données expérimentales et de leur interprétation de [4] dont j'ai choisi

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[4] B. R. Mottelson and S. G. Nilsson: *Det. Kong. Dan. Mat. Fys. Medd.*, **1**, 8 (1959).

trois transitions de ^{181}Ta, (Fig. 1), et une transition de ^{153}Eu, (Fig. 2), selon les critériums mentionnés plus haut. Ces transitions sont figurés ci-dessous.

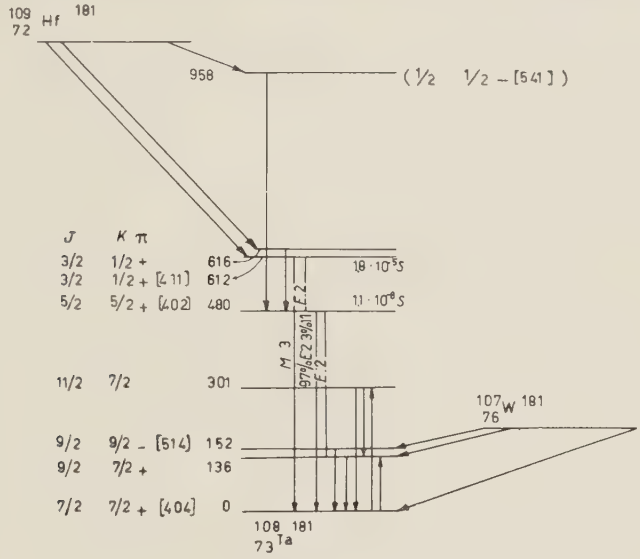

Fig. 1.

On a calculé les probabilités de transition autant dans la représentation approximative $(T_\delta(E2))$ que dans celle corrigée $(T_\varepsilon(E2))$. Les formules

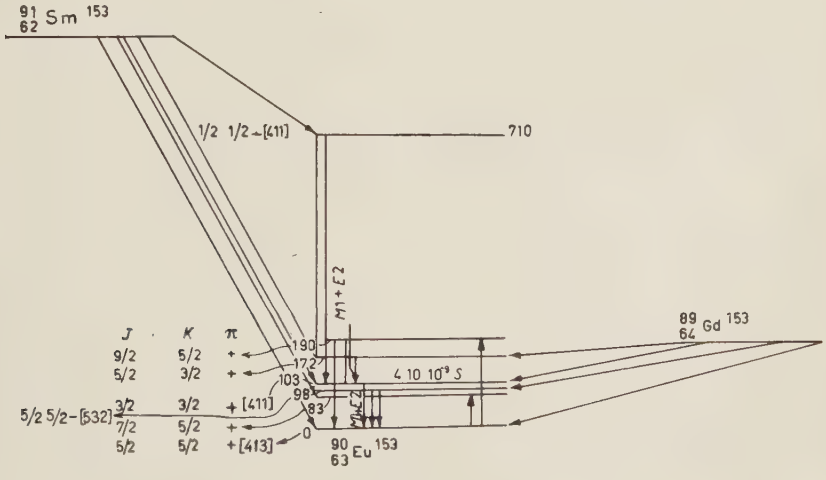

Fig. 2.

respectives écrites de manière a pouvoir y faire directement usage de toutes les données (expérimentales et théoriques) dont on dispose, sont

les suivantes

$$T_\delta(E2) = 0.2 \frac{[W_\gamma(\mathrm{keV})]^5}{[\hbar\overset{0}{\omega}_0(Q_0(\delta_e))\,(\mathrm{MeV})]^2} \cdot \left(1+\frac{Z}{A}\right)^2 \langle J_i 2K_i K_f - K_i | J_f K_f\rangle^2 G^2_{E2}(\delta_e)\cdot 10^{-4}\ \mathrm{s}^{-1},$$

$$T_\varepsilon(E2) = 0.2 \frac{[W_\gamma(\mathrm{keV})]^5}{[\hbar\overset{0}{\omega}_0(Q_0(\varepsilon_e))\,(\mathrm{MeV})]^2} \cdot \left(1+\frac{Z}{A}\right)^2 u(\varepsilon_e)\,\langle J_i 2K_i K_f - K_i | J_f K_f\rangle^2 G^2_{E2}(\varepsilon_e)\cdot 10^{-4}\ \mathrm{s}^{-1}.$$

Dans toutes les transitions considérées, les états initial et final ont $N = 4$. Le calcul de $T_\delta(E2)$ et $T_\varepsilon(E2)$ a été effectué autant au moyen des fonctions d'onde calculées pour $\mu = 0.45$, qu'au moyen de celles obtenues en considérant, comme dans [5], $\mu = 0.55$.

On a aussi calculé, à titre de comparaison, les probabilités de transition uniparticule dans l'hypothèse d'un potentiel self-consistant à symétrie sphérique ($J = K = \Omega = j$) et dans l'hypothèse de fonctions d'onde radiales constantes à l'intérieur du noyau. Si l'on tient compte aussi du facteur statistique, on peut écrire l'expression de cette probabilité comme

$$T_{s.s}(E2) \approx 0.7[W_\gamma(\mathrm{keV})]^5 A^{\frac{4}{3}}(2l_i+1)(2j_f+1)\langle l_i 200 | l_{f0}\rangle^2 W^2(l_i j_i k_f j_f ; \tfrac{1}{2}2)\ 10^{-7}\ \mathrm{s}^{-1}.$$

Les résultats obtenus sont donnés dans le Tableau suivant.

Noyau	$\hbar\overset{0}{\omega}_0$ utilisé	Transition	$T_\delta(E2)$ ($\mu=0.45$)	$T_\varepsilon(E2)$ ($\mu=0.45$)	$T_\delta(E2)$ ($\mu=0.55$)	$T_\varepsilon(E2)$ ($\mu=0.55$)	$T_{s.s.}(E2$
$^{181}_{73}$Ta ($\delta_e = 0.23$)	3.27 (MeV) (repr. appr.) 6.45 (MeV) (repr. corr.)	$(\frac{5}{2}+)\to(\frac{9}{2}+)$	$3.8\cdot10^6\ \mathrm{s}^{-1}$	$1\ \cdot10^6\ \mathrm{s}^{-1}$	$5.9\cdot10^6\ \mathrm{s}^{-1}$	$1.6\cdot10^6\ \mathrm{s}^{-1}$	$2.1\cdot10^8$ s
		$(\frac{5}{2}+)\to(\frac{7}{2}+)$	$2.4\cdot10^7\ \mathrm{s}^{-1}$	$6.8\cdot10^6\ \mathrm{s}^{-1}$	$4\ \cdot10^7\ \mathrm{s}^{-1}$	$1.1\cdot10^7\ \mathrm{s}^{-1}$	$1.6\cdot10^8$ s
		$(\frac{1}{2}+)\to(\frac{5}{2}+)$	$3.5\cdot10^3\ \mathrm{s}^{-1}$	$7.7\cdot10^2\ \mathrm{s}^{-1}$	$3.8\cdot10^3\ \mathrm{s}^{-1}$	$8.3\cdot10^2\ \mathrm{s}^{-1}$	$2.1\cdot10^5$ s
$^{153}_{63}$Eu ($\delta_s = 0.33$)	4.70 (MeV) (repr. appr.) 9.26 (MeV) (repr. corr.)	$(\frac{3}{2}+)\to(\frac{5}{2}+)$	$1.4\cdot10^1\ \mathrm{s}^{-1}$	$0.4\cdot10^1\ \mathrm{s}^{-1}$	$3\ \cdot10^3\ \mathrm{s}^{-1}$	$1.3\cdot10^3\ \mathrm{s}^{-1}$	$1.2\cdot10^5$ s

[5] R. B. Mottelson and S. G. Nilsson: *Phys. Rev.*, **99**, 1615 (1955).

On remarque dès l'abord que les grandeurs $T_\varepsilon(E2)$ et $T_\delta(E2)$, pour les deux valeurs du paramètre μ, diffèrent, le plus souvent, par un facteur ≈ 4. De même on remarque que les probabilités de transition dans les deux représentations sont plus grandes pour $\mu = 0.55$ que pour $\mu = 0.45$ et que les valeurs $T_{s.s}(E2)$ sont sensiblement plus grandes que les valeurs $T_\delta(E2)$ et $T_\varepsilon(E2)$ pour les deux valeurs de μ. On aurait pu s'attendre à ce que les valeurs $T_{s.s}(E2)$ fussent plus grandes que les valeurs $T_\delta(E2)$ ou $T_\varepsilon(E2)$, l'hypothèse de fonctions radiales constantes à l'intérieur du noyau étant une approximation très grossière, qui ne fait que conduire, pour la probabilité de transition, à une limite supérieure, même pour les noyaux peux déformés (*).

On peut essayer une confrontation avec l'expérience pour le cas des premières deux transitions qui détérminent en majeure partie la valeur τ du niveau initial. On obtient la valeur théorique de τ, plus appropriée que celle expérimentale ($\tau = 1.1 \cdot 10^{-8}$ s), à l'aide des valeurs $T_\varepsilon(E2)$ pour $\mu = 0.55$, égal a $5.3 \cdot 10^{-8}$ s, tandis que les valeurs $T_\varepsilon(E2)$ pour $\mu = 0.45$ conduisent à une période presque deux fois plus grande. La discrépance entre la valeur théorique et celle expérimentale de τ peut être expliquée, tout au moins partiellement, par la modification de la forme du noyau, due à la transition uniparticule, aussi bien que par le fait qu'il n'a pas été tenu compte des valeurs différentes du paramètre de déformation dans l'état initial et l'état final.

* * *

Je veux exprimer ma gratitude à M. le Professeur Şerban Ţiţeica pour la bienveillance avec laquelle il a suivi ce travail.

(*) Le retard des probabilités $T_\delta(E2)$ et $T_\varepsilon(E2)$ par rapport à $T_{s.s.}(E2)$ devient beaucoup plus grand si, selon l'usage, on considère le facteur statistique de $T_{s.s.}$ égal à 1.

Note ajoutée aux épreuves.

L'emploi de la représentation corrigée dans le calcul des probabilités $T(E2)$, des transitions γ qui ont lieu par la modification de la structure intrinsèque des noyaux, n'est pas aussi nécéssaire que dans le calcul des moments quadrupolaires. Les résultats ci-dessus ont été obtenus en tenant rigoureusement compte des hypothèses du modèle et des informations acquises par l'étude des moments quadrupolaires.

Evidemment on peut évaluer aussi ces probabilités independemment de ce schéma de calcul. Par exemple on peut utiliser une même valeur $\hbar\overset{0}{\omega}_0 = 41\, A^{-\frac{1}{3}}$ MeV dans les expressions de $T_\delta(E2)$ et $T_\varepsilon(E2)$. Dans ces conditions l'emploi de la représentation corrigée n'est pas nécéssaire dans tous les cas. Ainsi pour les transitions dont il est question

plus haut on obtient la plus grande différence entre les valeurs $T_\delta(E2)$ et $T_\varepsilon(E2)$ dans le cas de la transition $(\frac{3}{2}+) \to (\frac{5}{2}+)$ de $^{153}_{63}\mathrm{Eu}$ où pour $\mu = 0.55$, $T_\delta(E2) = 1.1 \cdot 10^3\ \mathrm{s}^{-1}$ et $T_\varepsilon(E2) = 1.6 \cdot 10^3\ \mathrm{s}^{-1}$.

Des calculs preliminaires montrent qu'une situation analogue paraît avoir lieu pour les moments d'inertie dans l'hypothèse du noyau rigide.

RIASSUNTO (*)

Si sono calcolate le probabilità delle transizioni γ di tipo $E2$ facendo uso della rappresentazione di Nilsson che tiene conto dell'accoppiamento fra stati con N diversi. A questo scopo si è fatto uso dei risultati ottenuti dal calcolo dei momenti elettrici quadripolari. Lo studio effettivo di alcune transizioni γ del tipo $E2$ di alcuni nuclei dispari nel campo $150 < A < 188$ dimostra che è necessario fare uso della suddetta rappresentazione.

(*) *Traduzione a cura della Redazione.*

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Pion-Momentum Spectrum from K^- Absorption in Helium.

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Summary. — The pion-momentum spectrum from K^--absorption stars in helium is calculated in the impulse approximation. The kaon is assumed to be pseudoscalar and to be captured from the 1*s* state. A final state consisting of free particles is used. The effect of final bound states is considered and taken into account phenomenologically. All final-state interactions are neglected. It is shown that the shape of the momentum spectrum is determined mainly by kinematics, *i.e.*, the shapes and position of the spectral peaks are quite insensitive to detailed dynamical assumptions such as the nature of the K parity and capture orbit. The effect of final-state interactions is discussed.

1. – Introduction.

It has recently been found that Λ^0's emerge from K^- absorption stars in helium in great profusion [1]. It is possible to estimate the expected rate of Λ^0 production from the reaction

$$(a) \qquad K^- + {}^4He \to \Lambda^0 + \pi + 3\gamma$$

(*) On leave from Syracuse University.

[1] J. Leitner, E. M. Harth, M. M. Block B. Brucker, I. Hughes, T. Kikuchi, C. Meltzer, A. Pevsner, F. Anderson, H. Cohn: *Bull. Am. Phys. Soc.* (New York Meeting, 1959).

on the basis of the K^--p absorption results [2] and charge independence. This expected rate is $\sim 8\%$; the observed rate is larger, by a factor of roughly 4. It would seem then that a large proportion of the emergent Λ^0's originates from K^- interactions which produce a Σ, with a subsequent Σ-nucleon interaction giving rise to a Λ^0 [3], according to the reaction

$$(b) \qquad K^- + {}^4He \to \Sigma + \pi + 3\mathcal{N}\,; \qquad \Sigma + \mathcal{N} \to \Lambda^0 + \mathcal{N}.$$

In order to estimate the relative contributions of (*a*) and (*b*) to the total Λ^0 production, the pion-momentum spectrum from K absorptions yielding neutral hyperons is studied. In the absence of internal motion, the momentum spectrum from (*a*) and (*b*) would show sharp spikes at 244 MeV/c and 160 MeV/c respectively. The initial-state internal motion, detailed dynamics of the process, etc., broaden the spectrum into two peaks of finite width. The purpose of this study is to calculate the spectrum explicitly and investigate its dependence on detailed dynamical assumptions.

The emergent pion spectrum of reactions (*a*) and (*b*) is calculated in the impulse approximation [4]. The kaon is taken to be spin 0 and pseudoscalar [5]. All final-state interactions are neglected; *s*-state K capture is assumed [6]. A final state consisting of free particles is used. The effect of final bound-nucleon states is considered and taken into account phenomenologically.

It is shown that the shape of the momentum spectrum is essentially determined by kinematics. The spectral peak widths are quite insensitive in shape and position to dynamical assumptions, *i.e.*, they are little affected by the relative importance of terms in the transition matrix or by details of the internal motion. The spectrum is likewise insensitive to the assumed K parity and capture orbit. The effect of final-state interactions is discussed.

2. – Description of calculation.

In the laboratory system, the matrix element M, for both processes (*a*) and (*b*),

$$K^- + {}^4He \to \pi + \left(\begin{matrix}\Lambda\\ \Sigma\end{matrix} + 3\mathcal{N}\right),$$

[2] W. Alvarez, H. Bradner, A. H. Rosenfeld, F. T. Solmitz and R. D. Tripp: *Nuovo Cimento*, **5**, 1026 (1956).

[3] This process has been discussed by Horwitz, Miller, Murray, Taft and Schwartz in connection with K^--d absorption: *Bull. Am. Phys. Soc.*, **3**, 363 (1958).

[4] A. Fuji and R. Marshak: NYO Report 2166, Univ. of Rochester (March 1957).

[5] A. Pevsner, F. Anderson, M. M. Block B. Brucker I. Hughes, T. Kikuchi, C. Meltzer, H. Cohn, E. Harth and J. Ceither: *Bull. Am. Phys. Soc.* (New York Meeting, 1959).

[6] Data from K^--p and K^--p absorption are consistent with pure *s* absorption.

is

$$M = \langle \chi_f | \int \psi_f^*(r_j, r_\pi)\, T(\boldsymbol{p}_j, \boldsymbol{k}, \boldsymbol{q}, \boldsymbol{\sigma}_1)\, \varphi_{nl}(|\boldsymbol{r}_K - \boldsymbol{R}|)\, \psi_i(r_j, r_K)\, \mathrm{d}\boldsymbol{r}_j \, | \chi_i \rangle ,$$

where: ψ_i = initial helium wave function,

ψ_f = final-state wave function,

φ_{nl} = K meson bound-state wave function,

T = a transition operator, the form of which depends upon the dynamic of the process,

$\chi_i \chi_f$ = appropriate spinors,

$\boldsymbol{r}_j$ = baryon lab. co-ordinate $j = 1, 2, 3, 4$,

$\boldsymbol{r}_\pi$ = pion lab. co-ordinate,

$\boldsymbol{r}_K$ = kaon lab. co-ordinate,

$\boldsymbol{k}$ = pion lab. momentum,

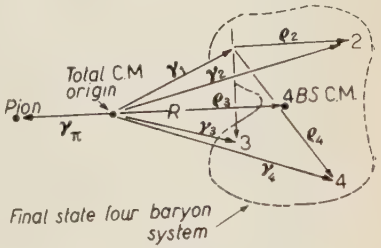
$\boldsymbol{q}$ = kaon lab. momentum,

$\boldsymbol{p}_j$ = center-of-mass momentum of baryon j in the final-state four-baryon system (4BS).

See Fig. 1 for a diagram of the co-ordinate sytems.

For the purposes of calculation, it is convenient to transform from $\boldsymbol{r}_j$ to a particular set of relative co-ordinates $\boldsymbol{\rho}_2$, $\boldsymbol{\rho}_3$, $\boldsymbol{\rho}_4$, $\boldsymbol{R}$ (where in accordance with the impulse approximation the K capture is presumed to take place on nucleon number 1. This transformation is defined by;

$$(1) \qquad \begin{cases} \boldsymbol{\rho}_2 = \boldsymbol{r}_2 - \boldsymbol{r}_1 , \\ \boldsymbol{\rho}_3 = \boldsymbol{r}_3 - \boldsymbol{r}_1 , \\ \boldsymbol{\rho}_4 = \boldsymbol{r}_4 - \boldsymbol{r}_1 , \\ \boldsymbol{R} = \dfrac{m_Y + m(\boldsymbol{r}_2 + \boldsymbol{r}_3 + \boldsymbol{r}_4)}{M} , \end{cases}$$



where: m = nucleon mass,

m_Y = hyperon mass, and

$M = 3m + M_Y$.

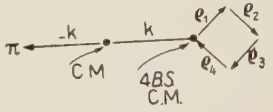


Fig. 1.

In terms of these variables, M becomes

$$(2) \qquad M = \langle \chi_f | \int \psi_f^*(\varrho_j)\, T(\boldsymbol{\rho}_j, \boldsymbol{q}, \boldsymbol{k}, \boldsymbol{\sigma})\, \varphi_{nl}\, \psi_i(\mathrm{d}\boldsymbol{\rho}_2\, \mathrm{d}\boldsymbol{\rho}_3\, \mathrm{d}\boldsymbol{\rho}_4) \, | \chi_i \rangle .$$

The ingredients of M are now considered in the order in which they appear above.

2·1. *The helium wave function.* – The helium wave function, ψ_i, is given by DALITZ [7]; ψ_i is taken as a Gaussian function of the relative co-ordinates and one parameter $\bar{\alpha}$, where $\bar{\alpha}$ is determined by the condition that the nucleon density fit the data of Hofstadter's electron-helium scattering experiment [8], yielding

$$\bar{\alpha} = \frac{9}{32 R_{\mathrm{He}}^2} . \qquad (R_{\mathrm{He}} = 1.44 \text{ fermi}).$$

In terms of the co-ordinate ϱ_j, we use

$$\psi_i = \exp\left(-\bar{\alpha} \sum_j \varrho_j^2\right), \qquad \text{for } j = 2, 3, 4. \tag{3}$$

2·2. *The transition operator and K wave function.* – The most general form of the transition operator T can be written in terms of vector products created from all vectors entering the problem; see, for example, the work of PAIS and TREIMAN [9]. The transformation properties of the vector products contained in T are determined by the condition that M be scalar. Since the pion is pseudoscalar, if the relative KΛ parity is odd, T must transform as a scalar. We assume here that this is indeed the case, although the final results are insensitive to this assumption. The most general scalar form of T is rather complex:

$$T = A + B \sum_j p_j^2 + Ck^2 + D \sum_j \boldsymbol{p}_j \cdot \boldsymbol{k} + E \sum_j \boldsymbol{\sigma} \cdot \boldsymbol{p}_j \times \boldsymbol{k} + \tag{4}$$

$$+ H \sum_j \boldsymbol{p}_j \cdot \boldsymbol{q} + J\, \boldsymbol{k} \cdot \boldsymbol{q} + K \sum_j \boldsymbol{\sigma} \cdot \boldsymbol{p}_j \times \boldsymbol{q} + L \boldsymbol{\sigma} \cdot \boldsymbol{k} \times \boldsymbol{q} .$$

This expression becomes considerably less formidable as a result of our physical approximations, described below.

T is an operator acting on the kaon wave function. Since in the impulse approximation, T is different from 0 only at $\boldsymbol{r}_1$ (the position of the nucleon at which the process is taking place), $T\varphi_{n1}$ must be evaluated at r_1. Physically, since the kaon co-ordinate corresponds to a Bohr orbit of radius $a_0 = 15 \cdot 10^{-13}$, *i.e.* 10 times the helium radius, this is equivalent to evaluating $T\varphi_{n1}(|r_{\mathrm{K}} - r_1|)$ at 0.

If we consider s-state capture only, terms in T independent of $\boldsymbol{q}$ give $\langle T\varphi_{1s}(0)\rangle \sim 1$, and terms in T proportional to $\boldsymbol{q}$ give $\langle T\nabla\varphi_{1s}(0)\rangle \sim 0$. We therefore drop all terms proportional to $\boldsymbol{q}$ in considering s-state absorption.

In the spirit of the impulse approximation, we further drop the terms

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(7) R. DALITZ: *Phys. Rev.*, **111**, 967 (1958).
(8) R. HOFSTADTER: *Rev. Mod. Phys.*, **28**, 214 (1956).
(9) A. PAIS and S. TREIMAN: *Phys. Rev.*, **107**, 1396 (1957).

proportional to $\boldsymbol{p}_i \cdot \boldsymbol{p}_j$, representing internal *correlations* within the 4BS, and obtain the relatively simple expression

$$|\langle T\varphi_{1s}(0)\rangle|^2 \sim 1 + ak^2 + b \sum_j p_j \mu_j , \tag{5}$$

where $\mu_j = \cos\theta_j$ and θ_j is the angle between $\boldsymbol{p}_j$ and $\boldsymbol{k}$. The undetermined constants a, b, c, representing detailed dynamics can be calculated only with further assumptions. No attempt to calculate a, b, and c is made here in view of the lack of a reliable dynamical theory of strong interactions ([10]).

2·3. *The final-state wave function.* – Neglecting final-state interactions (see discussion of results) the final-state wave function can be written as $\psi_f \sim \exp[(-i/\hbar)\boldsymbol{k}\cdot\boldsymbol{r}_\pi]\psi_{4\mathrm{BS}}(r_j, p_j)$. It has been shown that the hyperon emerges as a free particle in $\sim 99\%$ of all K interactions ([5]). The extent to which bound-nucleon states emerge from K stars is not well known yet, but is estimated to be small ([1]); ψ_f is therefore taken to be a free-particle-product wave function; with $\boldsymbol{r}_\pi$ evaluated at $\boldsymbol{r}_1$,

$$\psi_f = \exp\left[-\frac{i}{\hbar}\sum_j \boldsymbol{\rho}_j \cdot \left(\boldsymbol{p}_j - \frac{m}{M}\boldsymbol{k}\right)\right] . \tag{6}$$

3. – The general form of the rate.

Evaluation of the matrix element is straightforward ([11]). From (2) and (3) and (6) using $\alpha = \overline{\alpha}\hbar^2$, $\boldsymbol{k}_0 = (m/M)\boldsymbol{k}$ we obtain

$$R(k)\,\mathrm{d}k \sim [R_0(k) + aR_1(k) + bR_2(k) + cR_3(k)]\,\mathrm{d}k , \tag{7}$$

where

$$R_0 = k^2 \prod_j \left[\int \exp\left[-\frac{1}{2\alpha}(\boldsymbol{p}_j + \boldsymbol{k}_0)^2\right] p_j^2\,\mathrm{d}p_j\,\mathrm{d}\mu_j\,\delta(E - E_f)\right] ,$$

$$R_1 = k^2 R_0 ,$$

$$R_2 = k^2 \int \prod_j \left\{ \exp\left[-\frac{1}{2\alpha}(\boldsymbol{p}_j + \boldsymbol{k}_0)^2\right] p_j^2\,\mathrm{d}p_j\,\mathrm{d}\mu_j \right\} \sum_i p_i^2\,\delta(E - E_f) ,$$

$$R_3 = k^3 \int \exp\left[-\frac{1}{2\alpha}(\boldsymbol{p}_j + \boldsymbol{k}_0)^2\right] \sum_j p_j \mu_j \prod_j (\mathrm{d}\mu_j p_j^2\,\mathrm{d}p_j)\,\delta(E - E_f) .$$

([10]) FUJI and MARSHAK ([1]), using perturbation theory and global symmetry, have calculated analogous constants for K⁻-d absorption. The results do not agree with experiment.

([11]) This is done explicitly in a previous Berkeley Report UCRL-8747 (unpublished).

The integration of $R_j(k)$ is straightforward and the results are summarized as shown in Table I. The general form of $R_j(k)$ is

$$R_j(k) \sim k^r (k)^{s/2} \exp\left[-\frac{m}{\alpha}\left(f(k)+\Delta\right)\right], \tag{8}$$

where r and s are integers and

$$f(k) \equiv m_k + m - m_\Upsilon - B - \sqrt{k^2+\mu^2} - \frac{k^2}{2M},$$

$$\Delta = \left(\frac{3m}{M}\right)\frac{k^2}{2M}.$$

TABLE I. – *General form of the spectrum.*

$R_0(k)$	$= k^9 f(k)^{\frac{7}{2}} \exp\left[-m/\alpha[f(k)+\Delta]\right]$
$R_1(k)$	$= k^{11} f(k)^{\frac{7}{2}} \exp -\left[m/\alpha[f(k)+\Delta]\right]$
$R_2(k)$	$= k^{11}\left[f(k)^{\frac{9}{2}} + \frac{2}{33}\left(\frac{m}{\alpha^2}\right)^2\left(\frac{m}{M}\right)^4 f(k)^{11/2} k^4\right] \exp\left[-m/\alpha(f(k)+\Delta)\right]$
$R_3(k)$	$= k^9 f(k)^{\frac{7}{2}}\left[1 + \frac{\sqrt{2}m^{\frac{3}{2}}}{M\alpha}k + \frac{1}{3}\frac{m^{11}}{M^5\alpha^5}k^5 f(k)\right] \exp\left[-m/\alpha(f(k)+\Delta)\right]$

where

$$f(k) \equiv m_k + m - m_\Upsilon - B - \sqrt{k^2+\mu^2} - \frac{k^2}{2M},$$

$$\Delta \equiv \frac{3m}{M}\frac{k^2}{2M}.$$

For example, for $a=b=c=1$, one has

$$R(k) \sim \exp\left[-\frac{m}{\alpha}(f+\Delta\right] k^9 f^{\frac{7}{2}}\left[2 + k^2 + k^2 f + \frac{2}{33}\frac{m^6}{\alpha^2 M^4} f^2 k^6 + \frac{\sqrt{2}m^{\frac{3}{2}}}{M\alpha}k + \frac{1}{3}\frac{m^{11}}{M^5\alpha^5}k^5 f^2\right].$$

The zeroth order rate is shown in Fig. 2, with arbitrary normalization for the Σ and Λ peak. For comparison, the Λ peaks of both $R_0(k)$ and $R_1(k)$ are shown in Fig. 3. Mathematically, the similarity of all terms $R_j(k)$ is evident from (8). For low values of k, $R_j(k)$ is supressed by the factor k^r; $R_j(k)$ increases monotonically up to the region of the maximum, where its behavior is primarily determined by the exponential $\exp[(-m/\alpha) f(k)]$. Note that all $R_j(k)$

are proportional to this term. For very large values of k—, *i.e.*, $k \sim 0.95 k_{max}$ — the factor $f(k)^{s/2}$ rapidly damps the spectrum. Since all $R_j(k)$ are similar functions of k, the relative importance of each term—*i.e.* the relative magnitude of a, b, and c—does not seriously affect the total spectrum (7). $R(k)$ can thus be represented to a good approximation by the function $R_0(k)$ alone.

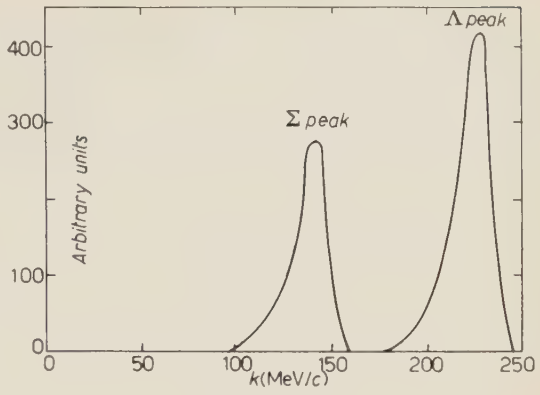

Fig. 2.

4. – Discussion of results.

The insensitivity of the spectrum $R(k)$, as calculated in the impulse approximation, to details of dynamics of both the initial state and the interaction can be easily understood on physical grounds. The effect of the Fermi-motion dynamics is apparent from the appropriate Fourier transform of the helium wave function:

$$(9) \qquad |\varphi(\theta_j p_j)|^2 \sim \prod_j \exp\left[-\frac{p_j k_0}{\alpha}\cos\theta_j\right] \prod_j \exp\left(-\frac{p_j^2}{2\alpha}\right) \sim$$

$$\sim \prod_j \exp\left[-2\cdot 10^{-3}\ p_j \cos\theta_j\right] \prod_j \exp\left[-10^{-4} p_j^2\right],$$

where $|\varphi|^2$ represents the distribution of internal momenta p_j and their c.m. angles θ_j which enter the problem. It is clear from the first exponential that $|\varphi|^2$ is insensitive to the magnitude of the angles θ_j. Thus the pion spectrum is insensitive to the probability distribution of θ_j, *i.e.*, to the detailed dynamics of the initial state. The second exponential in (9) shows that internal momenta $\leqslant 200$ MeV/c are of importance, this corresponds to an average nucleon velocity $\beta_{\mathcal{N}} \leqslant 0.2$, which *in the impulse approximation* is also the effective c.m. transformation velocity. Since the pion velocity ($\beta_\pi \sim 0.8$) is much larger than $\beta_{\mathcal{N}}$, the pion-momentum spectrum $R(k)$ is not seriously spread by the internal motion.

It should be stressed that this insensivity persists no matter what the form of the transition matrix, since the spectrum is mainly determined by the impulse-approximation kinematics in which K^- capture is considered a K^--$\mathcal{N}$ interaction.

It is not difficult to show explicitly that the assumptions of s-state capture and pseudoscalar kaon can be relinquished without any qualitative change in the pion spectrum.

We consider, firstly, the possibility of p-state capture. In this case, only terms in the transition matrix proportional to $\boldsymbol{q}$ will contribute to $R_{2p}(k)$, since

$$\varphi_{2p}(0) \sim 0 \qquad \text{and} \qquad \nabla\varphi_{2p}(0) \sim \text{constant} .$$

From (4), it follows that the $2p$-state matrix element is

$$|M_{2p}|^2 \sim [\mathrm{d}k^2 + e\sum p_j^2] \exp - \left[\frac{\sum_j (p_j + k_0)^2}{2\alpha}\right],$$

leading to essentially the same results as (7)

$$R_{2p'}(k) \sim dR_1(k) + eR_2(k) .$$

Secondly, we consider the possibility of a scalar K^-. In this case, the T matrix must transform as a pseudoscalar. The most general form for T is then

$$T \text{ (scalar } K) \sim g\boldsymbol{\sigma}\cdot\sum_j \boldsymbol{p}_j + h\boldsymbol{\sigma}\cdot\boldsymbol{k} + l\boldsymbol{\sigma}\cdot\boldsymbol{q} .$$

From (2) and (4) this leads directly to a rate

$$R(k)_{\text{scalar K}} \approx g^2 R_2(k) + h^2 R_1(k) + 2\mathrm{Re}\; g^* h R_3(k) ,$$

again, extremely similar to the spectrum (7).

It is clear from the above consideration that the pion-momentum spectrum is essentially determined by the kinematics of the K-$\mathcal{N}$ interaction and that its shape is insensitive to detailed dynamical assumptions within the framework of the impulse approximation.

The effect of bound-nucleon final-state production on $R(k)$ is essentially a kinematical one. The only kinematic difference between the free-particle and bound-state problem is the available kinetic energy in the final states. This difference appears in the energy δ function ([12]). In the bound-state problem $\delta(E - E_f)$ becomes $\delta(f(k) + V - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$, where V = final-state potential energy, $0 \leqslant V \leqslant 27$ MeV. To the extent, then, that internal correla-

tions in the final bound states are ignored, the problem reduces exactly to the free-particle problem, with the exception that $f(k)$ be replaced by $f(k)+V$.

Therefore $R_0^{\text{bound}}(k) \approx k^9[f(k)+V]^{\frac{7}{2}} \exp[(-m/\varkappa)(f(k)+V+1)]$. It is easily seen that in the production of bound final states; the peak position differs by only $\sim 0.6V$ MeV/c from its value in free-nucleon final-state production. Thus the qualitative features of the spectrum are not changed by the presence of bound final states.

The impulse-approximation theory of $\Lambda+\pi$ production described above is, without doubt, a severe oversimplification. The neglect of final-state interactions is, in particular, its most serious defect. This statement should be qualified; the omission of final-state *pion* interactions is not unreasonable. Pion-pion scattering is likely to be negligible because pion-nucleon interactions are important only near reasonance while the π-$\mathcal{N}$ c.m. energy is $\lesssim 70$ MeV in both reactions (*a*) and (*b*). On the other hand, the Σ is known to interact quite strongly, as evidenced by the importance of the Σ-Λ conversion process on the energy shell, thus Σ interactions *off* the energy shell, might well lead to a significant broadening of the emergent pion-momentum spectrum. An estimate of the importance of this effect has been made recently by KARPLUS and RODBERG [12] for K-deuterium interactions. They find that widely different pion spectral shapes are indeed possible. Since Σ-Λ conversion is at least as important in helium as it is in deuterium, it seems likely that this spectral broadening should also appear in K⁻-He absorption. The significant question whether the broadening is sufficient to vitiate the usefulness of the spectrum in distinguishing between direct and indirect Λ production cannot at this time be answered by further calculation, but must be referred to experiment.

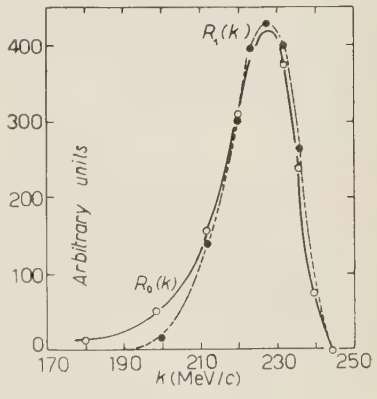

Fig. 3.

It can be hoped, in spite of its *naiveté*, that the simple approach outlined above may describe Λ production in K-He absorption at least qualitatively (this is the case in K⁻-D absorption). The fact that the calculated pion spectral shape is insensitive to all dynamical assumptions, excepting final-state Σ interactions, is an encouraging aspect of the approach in so far as it concerns the separation of direct and indirect Λ production.

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[12] B. KARPLUS and L. RODBERG: *Inelastic final state interactions* K⁻ *absorption in deuterium* (submitted to *Phys. Rev.*, 1959).

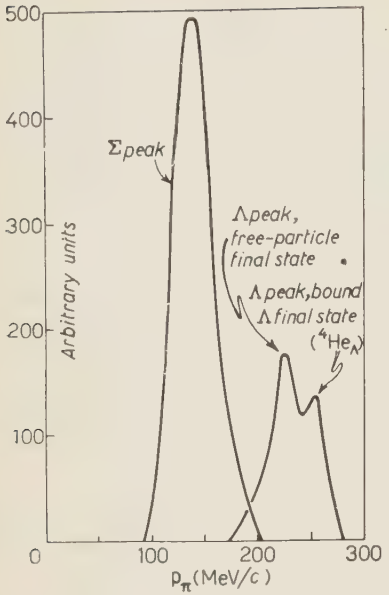

Fig. 4.

The impulse-approximation spectrum can be adequately represented by the function $R_0(k)$ shown in Fig. 3. In order to more readily compare with experiment, where momentum measurement is often limited to an accuracy of $\sim \pm 20$ MeV/c, we have folded $R(k)$ into an unnormalized experimental resolution function,

$$\exp\left[-\frac{1}{2}\left(\frac{p_\pi - k}{20}\right)^2\right].$$

The resulting spectrum, $F(p_\pi)$, is shown in Fig. 4. Preliminary experimental results [1] are in reasonable agreement with $F(p_\pi)$.

* * *

One of us (J.L.) should like to thank Drs. Stephen Gasiorowicz, Richard Arnowitt, and Donald Miller for several enlightening discussions and would also like to express his appreciation to Dr. Edward J. Lofgren and Dr. Luis Alvarez for their hospitality during his stay at Lawrence Radiation Laboratory.

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RIASSUNTO (*)

Si calcola lo spettro della quantità di moto dei pioni da stelle di assorbimento K^- in elio con l'approssimazione dell'impulso. Si suppone che il kaone sia pseudoscalare e che venga catturato dallo stato 1*s*. Si usa uno stato finale costituito da particelle libere. Si studia l'effetto degli stati legati finali e se ne tien conto fenomenologicamente. Si trascurano tutte le interazioni dello stato finale. Si dimostra che la forma dello spettro del momento è determinato principalmente dalla cinematica, cioè, le forme e le posizioni dei picchi spettrali non risentono per nulla da dettagliate supposizioni dinamiche quali la natura della parità del K e dell'orbita di cattura. Si discute l'effetto delle interazioni degli stati finali.

(*) *Traduzione a cura della Redazione.*

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Relativity and Quantum Theory.

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Summary. — It is shown that Einstein's generalized theory of gravitation as modified by the author contains integral and half integral spin fields. A constant of the dimension of a length plays a basic role in distinguishing between Boson and Fermion type fields.

1. – Introduction.

In this paper we propose a formal program for a theory of unified quantum physics. Einstein's general theory of relativity for gravitational and electromagnetic phenomena (in the absence of charges) can, classically, be regarded as a complete description of electromagnetic waves and their interactions with gravitational fields. In principle, general relativity has established a certain unity between symmetric (gravitational) and anti-symmetric (electromagnetic) fields. An electromagnetic field can act as a source of a gravitational field and therefore interact with it.

The interaction of a quantized electromagnetic field with a gravitational field is an example of weakest coupling of fields in nature. Generalization of electromagnetic and gravitational interactions may lead to other types of interactions. If we proceed in a historical order we observe that the photon, first introduced by Einstein and later described by a quantized electromagnetic field, has played an important role in the sequence of new discoveries in physics. In particular, de Broglie's theory of matter waves was inferred by the wave and particle behaviour of light. Electromagnetic waves do not carry mass and charge and they are described by a special type of an antisymmetric field (a gauge-invariant field or a field that is derivable from a vector potential). We may envisage a more general antisymmetric field, one that is not derivable from a vector potential. From a philosophical point of view

(the unity of natural phenomena) one is inclined to believe that the concept of generalized anti-symmetric field ought to play a basic role in describing both massless and mass fields. In most general terms a complete description of nature could be based on a certain geometrical union of symmetric and anti-symmetric fields. Einstein's general theory of relativity is a partial realization of this idea.

The existance of another massless field, the neutrino, poses a basic question since the neutrino is not described by an antisymmetric tensor field. We need spinors to describe a neutrino. Spinors themselves are not observables but we can construct observables from them. However, it is hard to justify to start the construction of a theory directly in terms of unobservable quantities. In this case an unlimited number of proposals can be made. Moreover, the mathematical nature of the quantities to be chosen for the formulation of physical laws and various transformation and symmetry properties of these laws require group theoretical jutification. In this paper we intend to be guided by classical analogs and by adherence to historical trend in physics.

The general theory of relativity (Einstein-Maxwell field equations without charges) must be an integral part of the unified theory, *i.e.*, it must be a limiting case of the theory. We shall see that mass and massless spinor fields will result in a natural way from such a theory. The intuitive perception of « waves » and of « particles » for photon and neutrino will lead to such properties for mass fields.

This paper will summarize some recent efforts on the author's version of Einstein's generalized theory of gravitation [1].

2. – General relativity.

In general relativity mass points are singularities of the field. General relativity, from the field point of view, can give a complete description only for the special case of a massless field. Its description of a field with mass is incomplete, since the field is not defined at the mass point itself.

The field equations follow from the variation of an action function

$$S_g = \frac{c^3}{16\pi G}\int L_g \, \mathrm{d}^4x \,, \tag{2.1}$$

where the Lagrangian L_g is given by

$$L_g = \sqrt{-a}\, a^{\varrho\sigma} G_{\varrho\sigma} + \frac{G}{c^4}\sqrt{-a}\, \varphi^{\varrho\sigma}(\varphi_{\varrho\sigma} - 2F_{\varrho\sigma}) \,, \tag{2.2}$$

[1] B. KURŞUNOĞLU: *Phys. Rev.*, **88**, 1369-1379 (1952).

G is the gravitational constant and the antisymmetric tensor $F_{\varrho\sigma}$ is derivable from a vector potential A_ϱ by

$$F_{\varrho\sigma} = \partial_\varrho A_\sigma - \partial_\sigma A_\varrho \,, \tag{2.3}$$

while $\varphi_{\varrho\sigma}$ is an arbitrary antisymmetric tensor and

$$a = \mathrm{Det}\,(a_{\varrho\sigma}) \,,$$

$$a^{\mu\varrho}\, a_{\mu\sigma} = \delta^\varrho_\sigma \,.$$

The action principle

$$\delta S_g = 0 \,,$$

with respect to the variation of the 20 independent variables $a_{\varrho\sigma}$, $\varphi_{\varrho\sigma}$, and A_ϱ leads to the field equations

$$\varphi_{\varrho\sigma} = F_{\varrho\sigma} \,, \tag{2.4}$$

$$\frac{\partial(\sqrt{-a}\,\varphi^{\varrho\sigma})}{\partial x^\sigma} = 0 \,, \tag{2.5}$$

$$G_{\varrho\sigma} = \frac{2G}{c^4}\, T_{\varrho\sigma} \,, \tag{2.6}$$

where

$$T_{\varrho\sigma} = \tfrac{1}{4} a_{\varrho\sigma}\varphi^{\mu\nu}\varphi_{\mu\nu} - \varphi_{\varrho\cdot}{}^{\mu}\varphi_{\sigma\mu} \,. \tag{2.7}$$

The electromagnetic field as the 4-dimensional curl of a 4-vector A_ϱ is thus incorporated into a variational principle.

The Lagrangian (2.2) can also be written as

$$L_g = \sqrt{-a}\,(a_{\varrho\sigma} + q^{-1}\varphi^{\varrho\sigma})(G_{\varrho\sigma} - p^2 q^{-1} F_{\varrho\sigma}) + 2p^2(\tfrac{1}{2}q^{-2}\Omega\sqrt{-a}) \,, \tag{2.8}$$

where

$$p^2 q^{-2} = \frac{2G}{c^4} \,, \tag{2.9}$$

$p^{-1} = r_0 =$ a constant of the dimensions of a length,

$q^2 =$ a constant of the dimensions of energy density,

$\Omega = \tfrac{1}{2}\varphi^{\varrho\sigma}\varphi_{\varrho\sigma}$.

Because of the relation (2.9) only one of the constants p and q is arbitrary. The actual physical meaning of the constant $p^{-1} = r_0$ will be taken up in the next section.

3. – Generalized theory of gravitation.

A purely geometrical derivation of the Lagrangian of the generalized theory of gravitation was given in reference (1). There is, however, another method which makes a direct use of the Lagrangian of general relativity written in the form of (2.8).

We may look for a variational principle in which the variables $a_{\varrho\sigma}$ and $q_{\varrho\sigma}$ are not treated independently. Such a possibility is also suggested from the form (2.8) of the symmetric theory. Instead of varying $a_{\varrho\sigma}$ and $q_{\varrho\sigma}$ separately we may propose to vary the 16 field variables $g_{\varrho\sigma}$ defined by

$$g_{\varrho\sigma} = a_{\varrho\sigma} + q^{-1}\varphi_{\varrho\sigma} . \tag{3.1}$$

In this case $g_{\varrho\sigma}$ as the fundamental field variables imply that the determinant of $a_{\varrho\sigma}$ in (2.8) must be replaced by

$$g = \mathrm{Det}\,(g_{\alpha\beta}) = a(1 + q^{-2}\,\Omega - q^{-4}\Lambda^2) , \tag{3.2}$$

where

$$\Lambda = \tfrac{1}{4} f^{\varrho\sigma}\varphi_{\varrho\sigma} = \text{pseudo-scalar} , \tag{3.3}$$

$$f^{\varrho\sigma} = \frac{1}{2\sqrt{-a}}\varepsilon^{\varrho\sigma\mu\nu}\varphi_{\mu\nu} . \tag{3.4}$$

The pseudo-scalar Λ as one of the two invariants of an anti-symmetric field (2) plays a basic role in the theory.

The term $a^{\varrho\sigma}+q^{-1}\varphi^{\varrho\sigma}$ in (2.8) should be replaced by $g^{\varrho\sigma}$ defined as

$$g^{\mu\varrho}g_{\nu\varrho} = \delta^{\mu}_{\nu} . \tag{3.5}$$

The generalization of the last term of (2.8) can be expected to involve both invariants Ω and Λ of the anti-symmetric field in the form

$$\sqrt{-g} - \sqrt{-a} = \sqrt{-a}\,\sqrt{1 + q^{-2}\Omega - q^{-4}\Lambda^2} - \sqrt{-a} , \tag{3.6}$$

which for large values of q reduces to $\frac{1}{2}q^{-2}\Omega\sqrt{-a}$.

The curvature tensor $G_{\varrho\sigma}$ of the symmetric theory is replaced by the curvature tensor

$$R_{\varrho\sigma} = R_{\underline{\varrho\sigma}} + q^{-1}R_{\underset{\sim}{\varrho\sigma}} \tag{3.7}$$

(2) Any anti-symmetric field $\varphi_{\varrho\sigma}$ contains an axial vector element. Such a symmetry property is a desiderable feature.

of the non-symmetric theory, where

$$R_{\mu\nu} = -\Gamma^{\varrho}_{\mu\nu,\varrho} + \Gamma^{\gamma}_{\mu\varrho}\Gamma^{\varrho}_{\gamma\nu} + \partial_{\mu}\partial_{\nu}(Ln\sqrt{-g}) - \Gamma^{\gamma}_{\mu\nu}\partial_{\gamma}(Ln\sqrt{-g}), \tag{3.8}$$

The action function of the generalized theory of gravitation is, therefore, given by

$$S(r_0) = \frac{q^2 p^{-2}}{8\pi c}\int L(r_0)\,\mathrm{d}^4x, \tag{3.9}$$

where

$$L(r_0) = \mathfrak{g}^{\mu\nu}(R_{\mu\nu} - p^2 q^{-1}F_{\mu\nu}) + 2p^2(\sqrt{-g} - \sqrt{-a}), \tag{3.10}$$

where p and q are related by (2.9)

It is of great importance to note that for $r_0 = 0$ we have

$$L(0) = L_g \tag{3.11}$$

and

$$S(0) = S_g . \tag{3.12}$$

The variation of (3.9) with respect to $g_{\mu\nu}$, A_{μ} and the affine connection $\Gamma^{\varrho}_{\mu\nu}$ lead to

$$R_{\underline{\mu\nu}} = -p^2(a_{\mu\nu} - b_{\mu\nu}), \tag{3.13}$$

$$R_{\underset{\smile}{\mu\nu}} = -p^2(\varphi_{\mu\nu} - F_{\mu\nu}), \tag{3.14}$$

$$\mathfrak{g}^{\underset{\smile}{\mu\nu}}{}_{,\nu} = 0, \tag{3.15}$$

$$g_{\mu\nu,\varrho} = g_{\mu\gamma}\Gamma^{\gamma}_{\varrho\nu} + g_{\gamma\nu}\Gamma^{\gamma}_{\mu\varrho} . \tag{3.16}$$

The equations (3.14) can also be cast in the form

$$R_{\underset{\smile}{\mu\nu},\varrho} + R_{\underset{\smile}{\nu\varrho},\mu} + R_{\underset{\smile}{\varrho\mu},\nu} = -p^2 I_{\mu\nu\varrho}, \tag{3.17}$$

where

$$I_{\mu\nu\varrho} = \varphi_{\mu\nu,\varrho} + \varphi_{\nu\varrho,\mu} + \varphi_{\varrho\mu,\nu} .$$

In carrying out the variation of (3.9) with respect to $g_{\mu\nu}$ the term $\sqrt{-a}$ in the Lagrangian will not cause any complication if we note that

$$\delta\sqrt{-a} = \tfrac{1}{2}a_{\mu\nu}\,\delta(\sqrt{-a}\,a^{\mu\nu}) = \tfrac{1}{2}b_{\mu\nu}\,\delta\mathfrak{g}^{\mu\nu}, \tag{3.18}$$

where

$$\mathfrak{g}^{\mu\nu} = \sqrt{-g}\,g^{\mu\nu} .$$

The symmetric tensor $b_{\mu\nu}$ is related to the symmetric tensor $a_{\mu\nu}$ by a similarity transformation

$$B = \tilde{K} A^{-1} K \,, \tag{3.19}$$

where the matrices B, K and A are defined by

$$B = (b_{\alpha\beta}) \,, \qquad A^{-1} = (a^{\alpha\beta}) \,, \qquad K = (K_{\alpha\beta}) \,,$$

$$a^{\alpha\varrho} a_{\beta\varrho} = \delta^{\alpha}_{\beta}$$

and

$$K_{\alpha\beta} = \frac{g_{\alpha\beta}}{(1 + q^{-2}\Omega - q^{-4}\Lambda^2)^{\frac{1}{2}}} \,,$$

so that

$$\operatorname{Det} \tilde{K} = \operatorname{Det} K = \operatorname{Det} A = a$$

and

$$\operatorname{Det} B = \operatorname{Det} A = a \,, \tag{3.20}$$

The metric tensor $b_{\alpha\beta}$ of the non-symmetric theory is given by

$$b_{\alpha\beta} = \frac{a_{\alpha\beta} + q^{-2}\varphi_{\alpha\varrho}\varphi_{\beta}^{\varrho}}{(1 + q^{-2}\Omega - q^{-4}\Lambda^2)^{\frac{1}{2}}} \,, \tag{3.21}$$

Throughout this paper we shall use the tensor $a_{\alpha\beta}$ for raising or lowering the tensor indices instead of the metric $b_{\alpha\beta}$.

The field equations (3.13), (3.14) and (3.15) of the generalized theory of gravitation reduces for $r_0 = 0$ to the field equations (2.4), (2.5) and (2.6) of general relativity [3].

The introduction of the constant r_0 is the main idea of this paper and it is the only deviation from Einstein's unified field theory [4]. It differentiates between massless and mass fields. Actually, it is the eigenvalue of our theory. The photon and neutrino fields are « zero length » gauge invariant fields. It is quite natural to expect that various values (eigenvalues) of the length r_0 will represent various mass fields defined by the solutions of the field equations.

The requirement

$$(1 + q^{-2}\Omega - q^{-4}\Lambda^2) = 0 \,, \tag{3.22}$$

---

[3] This statement is not always true since not all the solutions of the field equations would have a definite limit for $r_0 = 0$. Such solutions without the $r_0 = 0$ limit represent the classical counterpart of fermion fields. Actually the derivation of massless boson and fermion fields from the theory is based on certain types of gauge invariance properties. These points are discussed in Sections **7** and **8**.

[4] A. Einstein: *The Meaning of Relativity* (Princeton, 1953).

will put a limit to possible values of r_0 and finite values of r_0 can be expected to provide an upper limit for the maximum possible values of the field.

The square root $\sqrt{1+q^{-2}\Omega-q^{-4}\Lambda^2}$ is an invariant of the theory with very interesting properties. It will be found convenient to cast it in a more suggestive form. Expansion of the square root in powers of q^{-2} leads to

$$\sqrt{1+q^{-2}\Omega-q^{-4}\Lambda^2} = 1+\tfrac{1}{2}q^{-2}\Omega-\tfrac{1}{2}q^{-4}(\tfrac{1}{4}\Omega^2+\Lambda^2)+$$
$$+\tfrac{1}{4}q^{-6}\Omega(\tfrac{1}{4}\Omega^2+\Lambda^2)-\tfrac{1}{8}q^{-8}(\tfrac{1}{4}\Omega^2+\Lambda^2)^2-\tfrac{1}{8}q^{-8}\Omega^2(\tfrac{1}{4}\Omega^2+\Lambda^2)+\dots .$$

We note that from q^{-4} on all the terms contain the invariant $\frac{1}{4}\Omega^2+\Lambda^2$. It can be easily seen that the above expansion of the square-root implies that we can write it as

$$\sqrt{(1+\tfrac{1}{2}q^{-2}\Omega)^2-q^{-4}c^2p_\alpha p^\alpha}\,, \tag{3.23}$$

where

$$c^2p^\lambda p_\alpha = \tfrac{1}{4}\Omega^2+\Lambda^2\,, \tag{3.24}$$

By using the relations

$$T^\lambda_\mu T^\mu_\beta = \delta^\lambda_\beta(\tfrac{1}{4}\Omega^2+\Lambda^2)\,, \tag{3.25}$$

we can write

$$cp_\alpha = T^\varrho_\lambda V_\varrho\,, \tag{3.26}$$

where

$$V^\varrho V_\varrho = 1 \tag{3.k7}$$

is an arbitrary unit vector. From (3.25) and (3.27) it follows that the square root is invariant with respect to the introduction of the unit vector V_ϱ (*).

4. – Gauge invariance.

We must emphasize that our field variables $g_{\lambda\mu}$ with $r_0 \neq 0$ do not just represent a gravitational field plus an electromagnetic field (in the ordinary sense). This point has often been misunderstood and used by many physicists as a main objection to the theory on the basis that only electromagnetic and gravitational fields are included in the picture.

(*) In particular for $V^\varrho=(\mathrm{d}x^\varrho/\mathrm{d}s)$, $\mathrm{d}s^2=a_{\varrho\sigma}\mathrm{d}x^\varrho\,\mathrm{d}x^\sigma$, the invariance of the square-root with respect to the substitution (3.26) may be related to the motion of the eigenstates of the field.

This impression is certainly incorrect. Both symmetric and anti-symmetric parts of $g_{\nu\beta}$ are generalized field variables and are subject to the same geometrical restrictions imposed by the field equations.

EINSTEIN has pointed out that under a co-ordinate transformation symmetric and anti-symmetric parts of $g_{\nu\beta}$ transform separately; but under a gauge transformation associated with the affine connection of the field, *viz.*

$$\overset{*}{\Gamma}{}^{\varrho}_{\alpha\beta} = \Gamma^{\varrho}_{\alpha\beta} + \delta^{\gamma}_{\alpha}\frac{\partial\lambda}{\partial x^{\beta}}\,, \tag{4.1}$$

the field equations remain unchanged. Thus the symmetric and anti-symmetric parts of the connection transform together with respect to a one parameter gauge group. With respect to a one parameter transformation group we have a single field.

The potentials A_{ϱ} defined by (2.3) are subject to the same transformation (*i.e.* $A^{*}_{\varrho} = A_{\varrho} + \partial_{\varrho}\lambda$) so that the unobservable quantities A_{α} and $\Gamma^{\varrho}_{\alpha\beta}$ are related by gauge transformations. The electromagnetic field $F_{\nu\beta}$ is derivable from the potentials A_{ν} and the $R_{\nu\beta}$ field is derivable from the affine connections $\Gamma^{\varrho}_{\alpha\beta}$. However, while the potentials A_{ϱ} refer to a massless field, the connections $\Gamma^{\varrho}_{\nu\beta}$ are associated with mass fields. The latter is a consequence of the non-tensorial nature of the connections $\Gamma^{\varrho}_{\nu\beta}$. The gauge connection between A_{ν}'s and $\Gamma^{\varrho}_{\nu\beta}$'s points to the fact that massless and mass fields interact in a gauge invariant way (5).

5. – Spin and charge matrices.

We shall show in Section 7 that our theory contains half integral spin fields. In this section we give a brief discussion of spin and charge matrices. It is known that in a generally covariant theory the usual Dirac matrices are functions of space and time. We shall use a representation where (in the usua notation)

$$\boldsymbol{\gamma} = i\beta\boldsymbol{\alpha}$$

are hermitian and

$$\gamma_5^2 = -1\,, \tag{5.1}$$

$$\gamma_5 = \frac{1}{4!}\frac{1}{\sqrt{-a}}\,\varepsilon^{\mu\nu\varrho\sigma}\gamma_\mu\gamma_\nu\gamma_\varrho\gamma_\sigma\,, \tag{5.2}$$

$$\gamma_5\gamma_\mu + \gamma_\mu\gamma_5 = 0\,. \tag{5.3}$$

(5) A classical field theory need not have more to say on the role of gauge invariance in physics. The actual relation of mass to gauge trasformations of most general type can best be discussed within the scheme of quantum mechanics of this theory.

In unified theory there are two alternatives for connecting space and spin. We may either use the metric $b_{\nu\beta}$ or the symmetric part $a_{\nu\beta}$ of $g_{\nu\beta}$. If we use $a_{\nu\beta}$ for raising and lowering the tensor indices then we may also use it for connecting space and spin. In this case the γ-matrices satisfy the anti-commutation relations

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = -2a_{\mu\nu}I_4\,, \tag{5.4}$$

where I_4 is a 4-dimensional unit matrix.

Another relevant set of matrices refer to the generating matrices K_α of the three dimensional rotation group. The K-matrices are defined by

$$K_iK_j - K_jK_i = +iK_l\,, \tag{5.5}$$

$i\,j\,l = 1\,2\,3$ in cyclic order, and

$$K_4 = I_4\,,$$

$$K_\alpha^\dagger = K_\alpha\,.$$

The K-matrices defined in this way form a vector. In the more general case these matrices are functions of space and time. These matrices do not form a complete algebra. The matrices K_i $(i = 1, 2, 3)$ have eigen-values $+1$, -1 and 0. The latter property makes them most convenient for charge space representation, *i.e.* they can induce gauge transformations of the first kind. The explicit representation of K-matrices in the case of a flat space is given by

$$K_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{bmatrix} \quad K_2 = \begin{bmatrix} 0 & 0 & +i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \quad K_3 = \begin{bmatrix} 0 & i & -0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.6}$$

K_4 is a 3×3 unit matrix. They satisfy the relation

$$K_4^2 - K_1^2 - K_2^2 - K_3^2 = -1\,, \tag{5.7}$$

or in the case of general covariance we write it as

$$K_\varrho K^\varrho = -1\,. \tag{5.8}$$

In this case at each point of the field we have both spin and charge degree of freedom. These degrees of freedom change in a continuous fashion depending on a continuous group of co-ordinate transformations ([6]).

([6]) This novel feature of the theory does not imply a continuous distribution of charge since at each point the field defines K-matrices with eigen-values $+1$, -1 and 0.

6. – Extremum action and free particles in quantum theory.

In order to bring about the salient features of the proposals to be made and motivations in making these proposals for a new theory of matter we first discuss a few examples from classical and quantum mechanics.

A) The action function of a free non-relativistic particle in the time interval (t_0, t) is given by

$$S = \int_{t_0}^{t} \tfrac{1}{2} m v^2 \, \mathrm{d}t \,. \tag{6.1}$$

The equation of motion is

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = 0 \,.$$

The classical path of the particle is described by

$$\begin{cases} \boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{v}_0(t - t_0) \,, \\ \boldsymbol{v} = \boldsymbol{v}_0 \,, \end{cases} \tag{6.2}$$

so that

$$\boldsymbol{v} = \boldsymbol{v}_0 = \frac{\boldsymbol{r} - \boldsymbol{r}_0}{t - t_0} \,.$$

We use the path equations in (6.1) to calculate the extremum value of the action in the time interval (t_0, t) as

$$S = \tfrac{1}{2} m v_0^2 (t - t_0) = \frac{m(\boldsymbol{r} - \boldsymbol{r}_0)^2}{2(t - t_0)} \,, \tag{6.3}$$

along the path of the particle.

The S-function obtained in this way is a solution of the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S)^2 = 0 \,, \tag{6.4}$$

The extremum action S is the generator of a canonical transformation along the actual path of the particle. The corresponding situation in quantum mechanics is a canonical transformation along the « quantum paths » described by a transformation function defined by

$$\langle \boldsymbol{r}, t \,|\, \boldsymbol{r}_0, t_0 \rangle = \left[\frac{m}{2\pi i \hbar (t - t_0)} \right]^{\frac{3}{2}} \exp \left[\frac{i m (\boldsymbol{r} - \boldsymbol{r}_0)^2}{2\hbar (t - t_0)} \right] \,, \tag{6.5}$$

It satisfies the boundary condition

$$\lim_{t=t_0} \langle \boldsymbol{r}, t \,|\, \boldsymbol{r}_0, t_0 \rangle = \delta(r - \boldsymbol{r}_0)\,, \tag{6.6}$$

and the equation

$$\left(i\hbar \frac{\partial}{\partial t} - H_0\right) \langle \boldsymbol{r}, t \,|\, \boldsymbol{r}_0, t_0 \rangle = \delta(\boldsymbol{r} - \boldsymbol{r}_0)\, \delta(t - t_0)\,, \tag{6.7}$$

where

$$H_0 = -\frac{h^2}{2m} \nabla^2\,,$$

and

$$\int \langle \boldsymbol{r}, t \,|\, \boldsymbol{r}_0, t_0 \rangle \, \mathrm{d}^3\boldsymbol{r} = 1\,, \tag{6.8}$$

$$\Psi(\boldsymbol{r}, t) = \int \langle \boldsymbol{r}, t \,|\, \boldsymbol{r}', t' \rangle \, \Psi(\boldsymbol{r}', t') \, \mathrm{d}^3\boldsymbol{r}'\,. \tag{6.9}$$

In this case the quantum propagator of the free particle is not a function of the classical S alone but an additional phase factor is involved.

B) Relativistic free particle. – The classical action function S is given by

$$S = -mc^2 \int_{t_0}^{t} \sqrt{1 - \frac{v^2}{c^2}}\, \mathrm{d}t\,, \tag{6.10}$$

The extremum value of the action in this case is

$$\begin{cases} S = -mcR\,, \\ R = c^2(t - t_0)^2 - (\boldsymbol{r} - \boldsymbol{r}_0)^2 \end{cases} \tag{6.11}$$

and it satisfies the relativistic Hamilton-Jacobi equation

$$\left(\frac{\partial S}{\partial x_\varrho}\right)^2 + m^2c^2 = 0\,. \tag{6.12}$$

The corresponding quantum propagator of a free relativistic system satisfies the equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \varkappa^2\right) \Delta_F'(x - x') = -\delta(x - x')\,, \tag{6.13}$$

where the Δ_F function is the usual one expressed as a function of S in the form

$$\Delta^{(1)}(x-x') = \begin{cases} -\dfrac{\varkappa^2}{4\pi}\,\dfrac{N_1^{(1)}(1/\hbar)S}{(1/\hbar)S}, & S^2 > 0 \\ \dfrac{\varkappa^2}{4\pi}\,\dfrac{H_1^{(1)}(i/\hbar)(S)}{(1/\hbar)S}, & S^2 < 0 \end{cases}$$

and

$$\bar{\Delta}(x-x') = \frac{1}{4\pi}\,\delta\left(\frac{S^2}{m^2c^2}\right) - \frac{\varkappa^2}{8\pi}\left[\eta(x)+\eta(-x)\right]\frac{J\left((1/\hbar)S\right)}{(1/\hbar)S} \qquad \text{for all } S$$

$$\Delta_F(x-x') = \Delta^{(1)}(x-x') + \frac{2}{i}\,\bar{\Delta}(x-x')$$

and

$$\varkappa = \frac{mc}{\hbar}, \qquad \eta(x) \begin{cases} 1 & S^2 \geqslant 0, \quad t-t_0 < 0, \\ 0 & \text{otherwise}. \end{cases}$$

In this case the quantum propagator is constructed as a function of the classical action function alone.

C) Free Dirac particle. – In accordance with the two-valued representation of the Lorentz group we consider the relativistic extremum action function S defined by (6.11). The linearized form is given by

$$S = -mc\gamma^\varrho X_\varrho, \tag{6.14}$$

where

$$X_\varrho = x_\varrho - x'_\varrho.$$

The propagator of a free Dirac particle can, now, be written as

$$S_F(X) = \exp\left[-\frac{i}{\hbar}\,mc\gamma^\varrho X_\varrho\right]_{R=0} \Delta_F(x-x'), \tag{6.15}$$

where

$$\exp\left[-\frac{i}{\hbar}\,mc\gamma^\varrho X_\varrho\right]_{R=0} = \gamma_\varrho p_\varrho - \varkappa,$$

$$p_\varrho = -i\hbar\,\frac{\partial}{\partial x_\varrho}.$$

We thus see that in the last two examples (B and C) the classical extremum action function S plays a basic role in the construction of free particle quantum propagators.

7. – Extremum field action and particles.

The discussion of the previous section is not directly applicable to interacting systems. We do not know what particular role, if any, the extremum of an interacting classical system can play in the construction of propagators for interacting quantum mechanical systems. However, in unified theory it is not possible, actually it makes no sense, to speak of a free part and an interacting part. For the unified theory the definition of extremum field action can be given as the value of the action function obtained by substitution of the field equations in the expression of S from which the field equations were obtained by Hamilton's principle.

A) Boson fields. – We first consider the case of $r_0 = 0$ described by the action function (2.1). Substitution of the field equations (2.4) and (2.6) in (2.1) leads to an extremum action function

$$S_a = -\frac{1}{16\pi c}\int \sqrt{-a}\, F_{\varrho\sigma}F^{\varrho\sigma}\, \mathrm{d}^4x\,. \tag{7.1}$$

subject to the condition (2.5). It is the action of a pure radiation field in the presence of gravitation.

The extremum action function for $r_0 \neq 0$ fields is obtained from substituting the field equations (3.13) and (3.14) in (3.9). It gives

$$S(r_0) = -\frac{q^2}{4\pi c}\int (\sqrt{-g} - \sqrt{-a})\, \mathrm{d}^4x\,, \tag{7.2}$$

where we used the relations

$$g^{\varrho\sigma} = \sqrt{-b}\, b^{\varrho\sigma} = \sqrt{-a}\, b^{\varrho\sigma}\,, \qquad b^{\alpha\varrho}b_{\beta\varrho} = \delta^{\alpha}_{\beta}\,.$$

By using (3.22) we can write (7.2) as

$$S_b(r_0) = -\frac{q^2}{4\pi c}\int \sqrt{-a}\, \left\{\sqrt{(1+\tfrac{1}{2}q^{-2}\Omega)^2 - c^2 q^{-4} p^{\varrho}\bar{p}_{\varrho}} - 1\right\} d^4x\,, \tag{7.3}$$

This is the extremun action function of the unified theory and it reduces to (7.1) for $r_0 = 0$. The action (7.3) is, of course, subject to the four conditions imposed by the field equations (3.15).

B) Fermion fields. – In an infinitesimal region of space and time all points of the region can be connected by means of Lorentz transformations, regardless of the gravitational field. A two-valued representation of the Lorentz

group can be extended to a two-valued representation of the continuous group of co-ordinate transformations by means of gravitational potential dependent Dirac matrices. Now, our action (7.3) is a non-linear function of the 16 field variables $g_{\nu\beta}$ and therefore its rationalization will require the use of 16 Dirac matrices that constitute a complete (irreducible) matrix algebra in 4 dimensions. Let us assume the existence of an operator X defined by

$$X = F + B\gamma_5 + D_\nu\gamma^\alpha + L_\alpha\gamma_5\gamma^\alpha + M_{\alpha\beta}\gamma^\alpha\gamma^\beta ,$$

such that its square is an invariant given by

$$q^4\left[(1+\tfrac{1}{2}q^{-2}\Omega)^2 - c^2q^{-4}p_\nu p^\nu\right] ,$$

where F and B are scalars, D_ν and L_α vectors and $M_{\nu\beta}$ an antisymmetric tensor. The scalars F and B or the vectors D_ν and L_ν are because of the tensor property of our field variables $g_{\nu\beta}$ linearly independent. Now, the square of the operator X shows that there exist two possible operators whose squares fulfil the required condition. Hence the two action functions are given by

$$S_{f1}(r_0) = -\frac{1}{4\pi c}\int\sqrt{-a}\left[q^2(i\gamma_5 - 1) + \tfrac{1}{2}i\gamma_5\Omega + c\gamma^\varrho p_\varrho\right]\mathrm{d}^4x , \tag{7.4}$$

and

$$S_{f2}(r_0) = -\frac{1}{4\pi c}\int\sqrt{-a}\left[q^2(i\gamma_5 - 1) + \tfrac{1}{2}i\gamma_5\Omega + c\gamma_5\gamma^\varrho p_\varrho\right]\mathrm{d}^4x . \tag{7.5}$$

These action operators have resulted from (7.3) by taking in the expression of the operator X the values $F=0$, $M_{\alpha\beta}=0$, $L_\nu=0$, $B=i(1+\frac{1}{2}q^{-2}\Omega)$, $D_\alpha=cp_\alpha$ the action S_{f1} and $F=0$, $M_{\alpha\beta}=0$, $D_\nu=0$, $B_i=i(1+\frac{1}{2}q^{-2}\Omega)$, $L_\alpha=cp_\alpha$ for S_{f2}, respectively.

The expressions for S_{f1} and S_{f2} constitute an irreducible representation of the linearization process since i) there exists no canonical transformation to transform the γ_5 operator into a unit operator (the matrix γ_5 is a member of the complete algebra of 16 Dirac matrices in 4 dimensions); ii) because of the tensor property of $g_{\nu\beta}$ it is not possible to take a linear combination of the vectors D_ν and L_α, where $D_\alpha = L_\alpha$. But D_α and L_α can be taken equal as long as they occur in different expressions as in (7.4) and (7.5). Thus the two action operators (7.4) and (7.5) are not reducible into one another (7).

Linearization of an expression similar to (3.6) was attempted by Motz (8) for quite a different purpose, but his procedure was not invariant.

(7) Within the scheme of 4 dimensional matrix algebra the linearization of (7.3) is uniquely defined by (7.4) and (7.5). The possibility of higher order matrix algebra is, of course, not ruled out.

(8) L. Motz: *Phys. Rev.*, **89**, 60 (1953).

Now, the definition of p_λ by (3.26) shows that a tensor of rank two $(T_{\alpha\beta})$ is involved in the linearization process instead of a vector as in Dirac's theory. In the linearization of (7.3) γ-matrices alone are sufficient. Nevertheless, the arbitrary nature of the unit vector V_λ may offer some interesting possibilities.

For the sake of completeness we record the useful relations

$$\text{(7.6)} \qquad \mathrm{Det}\,(iq^2\gamma_5 + \tfrac{1}{2}i\gamma_5\Omega + c\gamma^\varrho p_\varrho) = \mathrm{Det}\,(iq^2\gamma_5 + \tfrac{1}{2}i\gamma_5\Omega + c\gamma_5\gamma^\varrho p_\varrho) = \\ = q^8(1 + q^{-2}\Omega - q^{-4}\Lambda^2)^2\,,$$

and

$$\text{(7.7)} \qquad \mathrm{Det}\,(T^\alpha_\beta) = (c^2 p_\lambda p^\lambda)^2\,.$$

8. – Mass and massless half integral spin fields.

We have seen that for boson fields the limit $r_0 = 0$ leads to a massless electromagnetic field in the presence of a gravitational field. For fermion fields the limit $r_0 = 0$ does not exist since the term q^2 in the action operators becomes infinite for the limit $r_0 = 0$. Thus there exist no limiting fermion fields with mass.

In order to understand mass states of the field we shall make use of the existence of the γ_5 operator in the action operators S_{f1} and S_{f2}. We introduce invariant «mass gauge operators» $\mathscr{M}_+$ and $\mathscr{M}_-$ by

$$\text{(8.1)} \qquad \mathscr{M}_\pm = \tfrac{1}{2}(1 \pm i\gamma_5)\,,$$

and write from (2.9) q^2 as

$$q^2 = \frac{1}{2}\frac{c^4}{Gr_0^2}\,.$$

The projection operators $\mathscr{M}_\pm$ have eigen-values $+1$ or 0. The action operators (7.4) and (7.5) can now be written in a suggestive form as

$$\text{(8.2)} \qquad S_{f1}(r_0) = -\frac{1}{4\pi c}\int\sqrt{-a}\left[c\gamma^\varrho p_\varrho + \tfrac{1}{2}\Omega - \mathscr{M}_-\left(\Omega + \frac{c^4}{Gr_0^2}\right)\right]\mathrm{d}^4x\,,$$

$$\text{(8.3)} \qquad S_{f2}(r_0) = -\frac{1}{4\pi c}\int\sqrt{-a}\left[c\gamma_5\gamma^\varrho p_\varrho + \tfrac{1}{2}\Omega - \mathscr{M}_-\left(\Omega + \frac{c^4}{Gr_0^2}\right)\right]\mathrm{d}^4x\,,$$

The association of the mass term (the r_0 term) with the projection operator $\mathscr{M}_-$ clearly shows the existence of massless and mass fermion fields in an invariant way (9).

(9) This new result of the theory is highly satisfactory. The relation between the γ_5 operator and the mass of fermion fields appears as a unified and invariant statement.

The existence of two different action operators for fermion fields may be interpreted as referring to two distinct classes of fermion fields. The two fermion classes differ in their symmetry properties.

9. – Quantum properties of fields.

Free massless fields (photon and neutrino) because of their description in terms of linear equations satisfy the principle of superposition. Quantization of linear fields is an experimental necessity. The mass fields obey highly non-linear equations. Quantization of non-linear fields is not an unambiguous concept and presumably is very difficult to interprete physically. Above all the execution of a quantization program for a non-linear field seems to be an impossible task.

The non-linearity of the fields arise from the union of all possible interactions of fields and also from the representation of mass by fields within a geometrical scheme. The non-linearity is a mathematical representation of matter fields and it is not directly quantized. The quantum properties of matter fields are induced by quantized massless free fields. The quantum behaviour of nature, in every respect, must result from the quantum properties of the two massless fields of photon and neutrino. These fields can be regarded as quantized sources of non-linear fields that are not quantized. If what we have said is correct we shall then expect to get solutions of non-linear field equations with quantum features.

The discussions of Section 6 seem to indicate that for free fields quantization can be carried out in such a way that the classical action function (its extremum value) plays a basic role in the construction of the field propagators. The use of the extremum value of the action instead of the action function itself differs from Feynman's [10] method since the former requires complete knowledge of the classical path.

In principle, with given extremum value of the classical action we can, for free fields, construct quantum propagators. For free electromagnetic fields the method is easy and quite straightforward; but for fermion fields the procedure is not quite obvious.

A somewhat different point of view is to introduce, instead of the operator form of the actions, the two invariant action functions

$$S_{\mathrm{I}}(r_0) = \int \Omega_{\mathrm{I}} \, \mathrm{d}^4 x \tag{9.1}$$

[10] R. P. Feynman: *Rev. Mod. Phys.*, **20**, 367 (1948).

and

$$S_{II}(r_0) = \int \mathfrak{L}_{II}\, \mathrm{d}^4x\,, \tag{9.2}$$

where

$$\mathfrak{L}_{I} = -\frac{1}{4\pi c}\langle\psi|\left|c\gamma^\mu p_\mu + \tfrac{1}{2}\Omega - \mathscr{M}_-\left(\Omega + \frac{c^4}{Gr_0^2}\right)\right||\psi\rangle\,, \tag{9.3}$$

$$\mathfrak{L}_{II} = -\frac{1}{4\pi c}\langle\eta|\left|c\gamma_5\gamma^\mu p_\mu + \tfrac{1}{2}\Omega - \mathscr{M}_-\left(\Omega + \frac{c^4}{Gr_0^2}\right)\right||\eta\rangle\,. \tag{9.4}$$

The kets $|\psi\rangle$, $|\eta\rangle$ are 4-component spinors and $\langle\overline{\psi}| = \langle\psi|\beta$, $\langle\overline{\eta}| = \langle\eta|\beta$. In this case the quantities $|\psi\rangle$ and $|\eta\rangle$ are to be regarded as the wave functions satisfying integro-differential equations resulting from the variations of S_I and S_{II} with respect to $|\psi\rangle$ and $|\eta\rangle$.

In principle, all predictions on the behaviour of a dynamical system can be made by calculating two classes of quantities: i) calculation of the field variables $g_{\alpha\beta}$, and hence of Ω, p_μ, etc., from the non-linear classical field equations (3.13), (3.14) and (3.15), ii) calculation of the wave functions $|\psi\rangle$ $|\eta\rangle$ from the wave equations. We need the wave functions to predict various reaction cross-sections of dynamical systems, atomic and nuclear energy levels, binding energies, etc.

Furthermore, the appearance of the projection operator $\mathscr{M}_-$ in the Lagrangian $\mathfrak{L}_I$ and $\mathfrak{L}_{II}$ can be interpreted as giving rise to mass degree of freedom of spin $\frac{1}{2}$ fields (since it is associated with the mass term). The eigen vectors of $\mathscr{M}_-$ corresponding to the eigen values 1, 0, 1, 0, are given by

$$|\mathscr{M}'_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\quad |\mathscr{M}'_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix},\quad |\mathscr{M}'_3\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\\0\end{bmatrix},\quad |\mathscr{M}'_4\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}, \tag{9.5}$$

where

$$\mathscr{M}_-|\mathscr{M}'_1\rangle = |\mathscr{M}'_1\rangle\,, \qquad \mathscr{M}_-|\mathscr{M}'_2\rangle = 0$$
$$\mathscr{M}_-|\mathscr{M}'_3\rangle = |\mathscr{M}'_3\rangle\,, \qquad \mathscr{M}_-|\mathscr{M}'_4\rangle = 0$$

and

$$\langle\mathscr{M}'_i|\mathscr{M}'_j\rangle = \delta_{ij}\,. \tag{9.6}$$

Thus the wave functions $|\psi\rangle$ and $|\eta\rangle$, in addition to spin co-ordinates, can have mass co-ordinates associated with them. In this sense the first two mass components of $|\psi\rangle$ correspond to mass and massless states of particles and the remaining two mass components correspond to mass and massless states

of anti-particles. This is equivalent to a simultaneous description of mass and massless particles and antiparticles, respectively (*).

Moreover, a more general concept of wave function follows from the complex representation of the Lorentz group (**).

The complex space representation of the momentum vector p_μ defined by (3.36) is given as

$$p_\mu = \frac{1}{2c}\langle\chi|B_{\mu\nu}V^\nu|\chi\rangle\,, \tag{9.7}$$

where the complex ket $|\chi\rangle$ in flat space-time has the form

$$|\chi\rangle = \begin{vmatrix} \varphi_{14}+i\varphi_{23} \\ \varphi_{24}+i\varphi_{31} \\ \varphi_{34}+i\varphi_{12} \end{vmatrix}$$

and the ten symmetric hermitian matrices are given by

$$\begin{cases} B_{44} = K_4 = I_3\,, \qquad B_{4i} = B_{i4} = K_i\,. \\ B_{ij} = B_{ji} = K_iK_j + K_jK_i - I_3\delta_{ij}\,. \end{cases} \tag{9.8}$$

In terms of $B_{\mu\nu}$-matrices the Maxwell tensor $T_{\mu\nu}$ can be expressed as

$$T = \tfrac{1}{2}\langle\chi|B_{\mu\nu}|\chi \quad . \tag{9.9}$$

With these premises the Lagrangians $\mathfrak{L}_{\text{I}}$ and $\mathfrak{L}_{\text{II}}$ can be written as

$$\mathfrak{L}_{\text{I}} = -\frac{1}{8\pi c}\langle\Phi|\left[B_{\mu\nu}\gamma^\mu V^\nu + Y - 2\mathscr{M}_-\left(Y + \frac{Yc^4}{\Omega G r_0^2}\right)\right]|\Phi \tag{9.10}$$

and

$$\mathfrak{L}_{\text{II}} = -\frac{1}{8\pi c}\langle\Theta|\left[B_{\mu\nu}\gamma_5\gamma^\mu V^\nu + Y - 2\mathscr{M}_-\left(Y + \frac{Yc^4}{\Omega G r_0^2}\right)\right]|\Theta\rangle\,, \tag{9.11}$$

(*) In other words, parity-conserving and parity-non-conserving states of spin $\frac{1}{2}$ fields appear, covariantly, as different components of a single wave function in the mass space spanned by the eigen-vectors of the projection operators $\mathscr{M}_-$. And, as a consequence of the four eigen vectors forming a complete set, all mass amd massless states are coupled. Under the time reversal operation the operator $\mathscr{M}_-$ is transformed into

$$\mathscr{M}_+ = \tfrac{1}{2}(1 + i\gamma_5)$$

and accordingly particle, and anti-particle mass and massless states are interchanged.

(**) A complex orthogonal and anti-orthogonal representation of the Lorentz group, by the author, is in course of publication and detailed discussion of the present section can be found in this latter paper.

where the generalized wave functions $|\Phi\rangle$ and $|\Theta\rangle$ are defined by

$$|\Phi\rangle = |\chi\rangle|\psi\rangle\,, \tag{9.12}$$

$$|\Theta\rangle = |\chi\rangle|\eta\rangle\,. \tag{9.13}$$

The hermitian operator Y is defined in terms of the anti-linear operators $\bar{C}$ and $\bar{C}_L$ as

$$Y = \tfrac{1}{2}(\bar{C} + \bar{C}_L)\,, \tag{9.14}$$

where the effect of the anti-unitary operator $\bar{C}$ is to replace the expression following it by its complex conjugate, while the effect of $\bar{C}_L$ is to replace the expression preceding it by its complex conjugate, in the complex space defined by the vectors $|\chi\rangle$. The hermitian operator Y does not operate on the 4-component spinor parts of the wave functions $|\Phi\rangle$ and $|\Theta\rangle$. From the operator properties

$$\bar{C}^2 = 1\,, \qquad \bar{C}_L^2 = 1\,, \qquad \bar{C}\bar{C}_L = \bar{C}_L\bar{C}$$

$$\bar{C}b = b\bar{C}_L = b^* \qquad \text{(where } b \text{ is a complex number)},$$

it follows that Y satisfies the algebraic equation

$$Y^3 = Y \tag{9.15}$$

and that

$$\langle\chi|Y|\chi\rangle = \Omega\,.$$

The operator Y commutes with the operators in the braces of (9.10) and (9.11). The eigen-values of Y are $+1$, -1, 0. The relation of the operator Y to the representation of charge space is discussed in connection with the complex representation of the Lorentz group.

Finally, we remark that the operator γ_5 appearing in the first term of the second action function, with the eigen-values $+i$ and $-i$ may be regarded as referring to relative phases of various fermion fields.

10. – Possible nature of the solutions of the field equations.

The results of Section 7 show that there exist two classes of extremum actions of the same field one referring to integral spin and the other to half integral spin type actions. In particular, we have seen that the boson type action function has for $r_0 = 0$ the electromagnetic field as its limiting field and fermion type actions have no limiting field with $r_0 = 0$. On this basis it is natural to infer that our field equations possess two groups of solutions; one

group of solutions having the electromagnetic field as a limit $r_0 = 0$ and another group of solutions having no limiting fields corresponding to $r_0 = 0$. If this conjecture proves to be right then bosons and fermions will come out to be integral and half integral spin states of a single field. Elementary particles will correspond to eigen-states of the field with the length r_0 as the eigen-value of the field.

The possibility of transitions from one eigenstate of the field into another and the duration of a particular eigenstate, the concept of nuclear force etc. are expected to result from the present theory of fields.

11. – Linear approximations.

All through this paper the importance of the non-linearity of the theory has been emphasized. Fortunately there is a simple way to justify the point. Consider the eigen-value equation

$$f_{\alpha\beta} u^{\beta} = f u_{\alpha} . \tag{11.1}$$

We use the identities

$$\varphi^{\alpha\mu} f_{\beta\mu} = \delta^{\alpha}_{\beta} \Lambda$$

and

$$\varphi^{\alpha\mu} \varphi_{\beta\mu} - f^{\alpha\mu} f_{\beta\mu} = \delta^{\alpha}_{\beta} \Omega$$

to obtain an eigen-value equation for the eigen-value f

$$f^4 - f^2 \Omega - \Lambda^2 = 0 . \tag{11.2}$$

The eigen value k of $\varphi_{\alpha\beta}$ is related to f by

$$k = -\frac{f}{\Lambda}$$

and it satisfies the equation

$$k^4 + k^2 \Omega - \Lambda^2 = 0 . \tag{11.3}$$

The roots of the equation (11.2) are $\pm f_1$ and $\pm f_2$, where

$$f_1 = \tfrac{1}{2} \left[\sqrt{\Omega + 2i\Lambda} + \sqrt{\Omega - 2i\Lambda} \right] ,$$

$$f_2 = \tfrac{1}{2} \left[\sqrt{\Omega + 2i\Lambda} - \sqrt{\Omega - 2i\Lambda} \right] .$$

Because of the pseudo-scalar character of Λ, f_2 is a pseudo-scalar and f_1 is a scalar. Corresponding to the four eigen-values $\pm f_1$, $\pm f_2$ there exist four eigen-

vectors. These linearly independent four vectors can span a four-dimensional invariant space. We have three cases to consider:

i) $\Lambda = 0$. In this case we can always find a reference system in which either the polar field (φ_{14}, φ_{24}, φ_{34}) or the axial field (φ_{23}, φ_{31}, φ_{12}) vanishes. The two linearly independent null-vectors can span an invariant subspace described by a single invariant parameter Ω. The mass density is given by

$$\frac{1}{c}\sqrt{p^\varrho p_\varrho} = \frac{\Omega}{2\ell^2}.$$

ii) $\Lambda = 0$, $\Omega = 0$. All four roots vanish. This subspace consists of a single point and is invariantly described by $\Omega = 0$, $\Lambda = 0$. The density of mass is zero.

iii) $\Lambda \neq 0$ and $\Omega \neq 0$. In this case it is not possible to find a reference system where the field is purely polar or axial. All four roots differ from zero and therefore there exist four linearly independent eigen-vectors spanning a 4 dimensional invariant space. The mass density is given by

$$\frac{1}{c}\sqrt{p_\varrho p^\varrho} = \frac{1}{c^2}\sqrt{\tfrac{1}{4}\Omega^2 + \Lambda^2}.$$

We see that both invariants Ω and Λ play an important role in the classification of mass space. In a linear approximation the invariant Λ appears for the first time in the fourth order approximation in powers of r_0.

Now, let us suppose that a linear approximation as in the theory of gravitation is possible. We may then expand the field variables in powers of r_0 as

$$g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + r_0 g^{(1)}_{\alpha\beta} + r_0^2 g^{(2)}_{\alpha\beta} + \dots . \tag{11.4}$$

It has been shown by HLAVATY [11] that if one neglects the terms r_0^2, r_0^3, ..., then one *a priori* excludes the cases i) and iii) discussed in the above. Thus a linear approximation is effectively valid (and only for boson fields) for the case ii). One has to include at least terms of order r_0^4 to produce the effects involved in i) and iii) to a first approximation. Thus the linear approximations exclude the possibility of deriving the fermion fields from the theory and also do not allow solutions with non-vanishing mass.

12. – Spherically symmetric fields.

The simplest possible non-linear field is the spherically symmetric field. Unfortunately, despite the great symmetry and simple structure of the sphe-

[11] V. HLAVATY: *Geometry of Einstein's Unified Field Theory* (Groningen, 1957), p. 14.

rically symmetric field, it has not yet been possible to solve the corresponding field equations ([12]).

PAPAPETROU ([13]) has shown that the spherically symmetric $g_{\alpha\beta}$ in polar co-ordinates can be brought into the form

$$(12.1)\qquad (g_{\varrho\sigma}) = \begin{bmatrix} -\alpha & 0 & 0 & \delta \\ 0 & -\beta & \chi\sin\theta & 0 \\ 0 & -\chi\sin\theta & -\beta\sin^2\theta & 0 \\ \delta & 0 & 0 & \mu \end{bmatrix},$$

where

$$\chi = r^2 A$$

and

$$A = \text{axial field}.$$

We may think of four possible spherically symmetric fields: *a*) both A and δ vanish, resulting in a pure gravitational field; *b*) $A = 0$, so that $\Lambda = 0$ ((this entails the case i)); *c*) $\delta = 0$ (this case also is included in i)); and *d*) none of Λ and δ vanish. The case *d*) is the most general spherically symmetric field that depends on both parameters Λ and Ω. The cases *a*), *b*) and *c*) have some degree of arbitrariness in them. The choice of *b*), *c*) or *d*) for consideration is, of course, greatly affected by the mathematical machinery of the problem. The actual physical problem is posed by the case *d*). However, for a first orientation we choose to investigate some of the properties of the case *c*). In this case the only non-vanishing components of the metric tensor $b_{\alpha\beta}$ and of the field $\sqrt{-g}\,g^{\alpha\beta}$ are given by

$$(12.2)\qquad \left\{\begin{aligned} b_{11} &= -\frac{\alpha\beta}{\sqrt{\beta^2 + q^{-2}\chi^2}}, \\ b_{22} &= -\sqrt{\beta^2 + q^{-2}\chi^2}, \\ b_{33} &= b_{22}\sin^2\theta, \\ b_{44} &= \frac{\mu\beta}{\sqrt{\beta^2 + q^{-2}\chi^2}}, \\ \sqrt{-a} &= \beta\sqrt{\alpha\mu}\sin\theta, \\ g^{23} &= \frac{\chi\sqrt{\alpha\mu}}{\sqrt{\beta^2 + q^{-2}\chi^2}}. \end{aligned}\right.$$

([12]) This is hardly surprising since many results are hoped to come with these solutions of the field equations.

([13]) A. PAPAPETROU: *Proc. Roy. Irish. Ac.*, **52**, A 6, 69 (1948).

The affine connections $\Gamma^{\nu}_{\alpha\beta}$ satisfying the algebraic equations (3.16) in the spherical symmetric case are given by

$$\Gamma^1_{11} = \frac{1}{2}\frac{\alpha'}{\alpha}, \qquad \Gamma^4_{14} = \frac{1}{2}\frac{\mu'}{\mu}, \qquad \Gamma^1_{44} = \frac{1}{2}\frac{\mu'}{\alpha},$$

$$\Gamma^2_{\underline{12}} = \Gamma^3_{\underline{13}} = \tfrac{1}{2}[\ln\sqrt{\beta^2 + q^{-2}\chi^2}],$$

$$\Gamma^1_{33} = \Gamma^1_{22}\sin^2\theta, \qquad \Gamma^3_{\underline{23}} = \operatorname{ctg}\theta, \qquad \Gamma^2_{33} = -\sin\theta\cos\theta,$$

$$\Gamma^1_{22} = \frac{1}{2\alpha}\left[q^{-1}\chi\left(\operatorname{tg}^{-1}\frac{\beta q}{\chi}\right)' - \beta\,(\ln\sqrt{\beta^2 + q^{-2}\chi^2})'\right],$$

$$\Gamma^1_{\underset{\smile}{23}} = \frac{\sin\theta}{2\alpha}\left[\beta\left(\operatorname{tg}^{-1}\frac{\beta q}{\chi}\right)' + q^{-1}\chi\,(\ln\sqrt{\beta^2 + q^{-2}\chi^2})'\right],$$

$$\Gamma^2_{\underset{\smile}{31}} = -\frac{1}{2}\left(\operatorname{tg}^{-1}\frac{\beta q}{\chi}\right)'\sin\theta,$$

$$\Gamma^3_{\underset{\smile}{12}} = \frac{1}{\sin^2\theta}\,\Gamma^2_{\underset{\smile}{31}},$$

where

$$F' = \frac{\mathrm{d}F}{\mathrm{d}r}.$$

The non-vanishing components of $R_{\alpha\beta}$ are given by

$$R_{11} = \frac{1}{v}(v\varrho')' + \tfrac{1}{2}(\varrho'^2 + \varphi'^2) + \tfrac{1}{2}u'[u + \varrho + \ln(u'v)]',$$

$$R_{22} = -\tfrac{1}{2}v\,[ve^{u+\varrho}(\varphi'\cos\varphi - \varrho'\sin\varphi)]' - \tfrac{1}{2}\varphi' v^2 e^{u+\varrho}(\varphi'\sin\varphi + \varrho'\cos\varphi) - 1,$$

$$R_{33} = R_{22}\sin^2\theta,$$

$$R_{44} = \tfrac{1}{2}(u'u^{2u}v^2)' - \tfrac{1}{2}ue^{2u}v^2\left(\varrho' - \frac{v'}{v}\right) + \tfrac{1}{2}u'^2v^2e^{2u},$$

$$\frac{1}{\sin\theta}R_{\underset{\smile}{23}} = -\tfrac{1}{2}v\,[ve^{u+\varrho}(\varphi'\sin\varphi + \varrho'\cos\varphi)]' + \tfrac{1}{2}v^2\varphi' e^{u+\varrho}(\varphi'\cos\varphi - \varrho'\sin\varphi).$$

The field equations (3.15) are automatically satisfied. The field equations (3.17) can be integrated once as

$$\frac{1}{\sin\theta}R_{\underset{\smile}{23}} = Q - p^2q^{-1}\chi, \tag{12.3}$$

where Q is a constant of integration. In the above we used the transformations

$$\text{(12.4)} \qquad \begin{cases} q = \operatorname{tg}^{-1}\left(\dfrac{\beta_q}{\chi}\right), \\ \varrho = \ln\sqrt{\beta^2 + q^{-2}\chi^2}, \\ e^u = \mu, \\ \dfrac{e^{-u}}{r^2} = \varkappa. \end{cases}$$

The constant of integration in (12.3) is a dimensionless constant. The constant Q cannot be a function of the constants c, G, r_0 alone since it is not possible to construct a dimensionless constant out of c, G and r_0. We need, at least one more constant to form a dimensionless constant. The nature of the latter constant can easily be decided. Let us suppose that the constant Q is not zero and that for $r \to \infty$ the function R_{23} tends to zero. In this case the function χ must tend to a constant of the dimensions of electric charge in such a way that the right hand side of (12.3) vanishes, *viz.*

$$\text{(12.5)} \qquad Q = p^2 q^{-1} \varepsilon,$$

where ε is a constant of the dimensions of an electric charge. How do we find the limit of the function χ? or what can we find out about the constant ε? First, we must point out that the theory cannot create another universal or ordinary constant without providing some experimental facts. Furthermore the theory is self-consistent *i.e.* fields are sources of the field and the constant r_0 is defined by the field and the field itself is defined as a function of r_0. But the field does not define the value of the constant. The information we are seeking can only be provided by quantizing the $F_{\nu\mu}$ field (the free electromagnetic field). In this way the missing universal constant h can be introduced in a most natural way. There are at least two more constants of integration to come. Their identification together with the eigen-value r_0 of the theory will not present any problem since we have enough constants to take care of, all measurable physical quantities.

By using the result (12.5) we can write the equation (12.3) as

$$\text{(12.6)} \qquad \frac{1}{\sin\theta} R_{23} = p^2 q^{-1}(\varepsilon - \chi).$$

We may now inquire about the limiting field $r_0 = 0$. For $r_0 = 0$ the right hand side of (12.6) is indeterminate. The limit can only be defined by the types

of solutions of the field equations. One expects to find a definite limit in the case of boson type solutions and no limit in the case of fermion solutions.

By suitable linear combinations of the various components of $R_{\nu\beta}$ the field equations (3.13) and (3.14) can be written in a neat form. We form the combinations

$$\beta R_{\underset{\sim}{23}} + q^{-1}\chi \sin\theta\, R_{22} = p^2 q^{-1}\beta \sin\theta - p^2 q^{-1}\chi \sqrt{\beta^2 + q^{-2}\chi^2} \sin\theta\,,$$

$$q^{-1}\chi R_{\underset{\sim}{23}} - \beta \sin\theta\, R_{22} = p^2 q^{-2}\chi(\varepsilon - \chi) - p^2\beta \sin\theta(\beta - \sqrt{\beta^2 + q^{-2}\chi^2})\,,$$

$$R_{11} - \frac{\alpha}{\mu} R_{44} = 2p^2\alpha \left| 1 - \frac{\beta}{\sqrt{\beta^2 + q^{-2}\chi^2}} \right|,$$

$$R_{11} + \frac{\alpha}{\mu} R_{44} = 0\,.$$

Hence the spherically symmetric field equations are

$$\tfrac{1}{2}v(ve^{u+\varrho}\varphi')' = p^2 e^{\varrho}\cos\varphi - p^2 q^{-1}\varepsilon \sin\varphi - \cos\varphi\,, \tag{12.7}$$

$$\tfrac{1}{2}v(ve^{u+\varrho}\varrho')' = p^2 e^{\varrho}(1 - \sin\varphi) - p^2 q^{-1}\varepsilon \cos\varphi + \sin\varphi\,, \tag{12.8}$$

$$\tfrac{1}{2}v(ve^{u+\varrho}u')' = p^2 e^{\varrho}(1 - \sin\varphi)\,, \tag{12.9}$$

$$\varrho'' + \varrho'\frac{v'}{v} + \frac{1}{2}(\varphi'^2 + \varrho'^2) = 0\,. \tag{12.10}$$

Apart from a few obvious observations on the general mathematical structure of the equations (12.7)–(12.10) it has not yet been possible to find the solutions of these equations. The equation (12.10) can be written as

$$\tfrac{1}{2}v(ve^{u+\varrho}\varrho')' = \tfrac{1}{4}v^2 e^{u+\varrho}(\varrho'^2 - \varphi'^2 + 2u'\varrho')\,.$$

Comparing with (12.8) we get the variable v as

$$v = 2\sqrt{Y}\exp[-\tfrac{1}{2}(u + \varrho)](\varrho'^2 - \varphi'^2 + 2\varrho' u')^{-\frac{1}{2}}, \tag{12.11}$$

where

$$Y = p^2 e^{\varrho}(1 - \sin\varphi) - p^2 q^{-1}\varepsilon \cos\varphi + \sin\varphi\,.$$

Two useful identities are

$$\begin{cases} X\cos\varphi - Y\sin\varphi - Z = -1\,, \\ Y' = \varrho' Z - \varphi' X\,, \end{cases} \tag{12.12}$$

where

$$X = p^2 e^{\varrho} \cos\varphi - p^2 q^{-1}\varepsilon \sin\varphi - \cos\varphi\,,$$

$$Z = p^2 e^{\varrho}(1 - \sin\varphi)\,.$$

In the limit $r_0 = 0$ the equation (12.7) does not lead to a definite limit but the equations (12.8), (12.9) and (12.10) reduce to the spherically symmetric field of general relativity with electromagnetic field. In this limit the variable φ tends to $(n+\frac{1}{2})\pi$, where $n = 0, 1, 2, \ldots$. The appearance of the discrete number $(n+\frac{1}{2})$ arises from the functional dependence of the affine connections on $\mathrm{tg}^{-1}(\beta q/\chi)$.

13. – Energy tensor of the field.

In general relativity, as a result of general covariance, there exists, independently from the field equations, an energy tensor and it is conserved everywhere. The same applies to unified field theory. The energy tensor $\mathfrak{T}_{\beta}^{\lambda}$ is conserved everywhere

$$\mathfrak{T}^{\alpha}_{\beta,\alpha} = 0\,, \tag{13.1}$$

where

$$-4\pi p^2 q^{-2}\mathfrak{T}^{\alpha}_{\beta} = \tfrac{1}{2}[\mathfrak{g}^{\alpha\mu}R_{\beta\mu} + \mathfrak{g}^{\mu\lambda}R_{\mu\beta} - \delta^{\alpha}_{\beta}\mathfrak{g}^{\mu\nu}R_{\mu\nu}] + \tfrac{1}{2}[\mathfrak{g}^{\mu\nu}{}_{,\beta}B^{\alpha}_{\mu\nu} - \delta^{\alpha}_{\beta}B]\,, \tag{13.2}$$

is the gauge invariant energy tensor of the field.

We use the expression

$$R_{\alpha\beta} = B^{\gamma}_{\alpha\beta,\gamma} + B_{\alpha\beta}$$

and the equations

$$\mathfrak{g}^{\alpha\beta}_{+\,;\gamma} = 0$$

to write $\mathfrak{T}^{\alpha}_{\beta}$ as

$$\mathfrak{T}^{\alpha}_{\beta} + \frac{c^4}{16\pi G}\frac{\partial S^{\alpha\varrho}_{\beta}}{\partial x^{\varrho}} = 0\,, \tag{13.3}$$

where

$$B^{\gamma}_{\alpha\beta} = \delta^{\gamma}_{\alpha}\Gamma^{\varrho}_{\beta\varrho} - \Gamma^{\gamma}_{\alpha\beta}\,,$$

$$B_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\mu}\Gamma^{\mu}_{\gamma\beta} - \Gamma^{\gamma}_{\alpha\beta}\Gamma^{\varrho}_{\gamma\varrho}\,,$$

$$S^{\alpha\varrho}_{\beta} = \mathfrak{g}^{\lambda\mu}B^{\varrho}_{\beta\mu} + \mathfrak{g}^{\mu\alpha}B^{\varrho}_{\mu\beta} - \delta^{\alpha}_{\beta}\mathfrak{g}^{\mu\nu}B^{\varrho}_{\mu\nu}\,.$$

The trace of $\mathfrak{T}^{\lambda}_{\beta}$ is given by

$$\mathfrak{T} = \mathfrak{T}^{\lambda}_{\lambda} = \frac{\partial \mathfrak{H}^{\lambda}}{\partial x^{\lambda}}\,,$$

where

$$\mathfrak{U}^{\alpha} = \frac{c^4}{8\pi G}\,\frac{1}{\sqrt{-g}}\,\frac{\partial}{\partial x^{\beta}}\,(\sqrt{-g}\,\mathfrak{g}^{\alpha\beta}). \tag{13.4}$$

At every point of the field we can define a generalized momentum vector by a surface integration of (13.3),

$$P_{\alpha} + \frac{c^4}{16\pi G}\int \frac{\partial S_{\alpha}^{\nu\varrho}}{\partial x^{\varrho}}\,\mathrm{d}\sigma_{\nu} = 0\,, \tag{12.5}$$

where $\mathrm{d}\sigma_{\nu}$ is a 4-dimensional surface element.

The use of the field equations in the energy tensor (13.2) leads to the result

$$\mathfrak{T}_{\beta}^{\alpha} = \frac{1}{4\pi}\left[q^{-2}(\sqrt{-g} - \sqrt{-a})\delta_{\beta}^{\alpha} - \mathfrak{g}^{\alpha\mu}F_{\beta\mu}\right] + \frac{1}{2}\mathfrak{T}\delta_{\beta}^{\alpha} - \frac{c^4}{16\pi G}\,\mathfrak{g}^{\mu\nu}{}_{,\beta}\,B_{\mu\nu}^{\alpha} \tag{13.6}$$

which reduces to general relativistic energy tensor in the limit $r_0 = 0$.

14. – Conclusion.

The strength of the coupling of the electromagnetic field to the gravitational field is measured by the constant

$$g_E = \frac{2G}{c^4} = p^2q^{-2} = 1.65\cdot 10^{-49}\ \mathrm{erg^{-1}\cdot cm}\,.$$

This is about of the same order of magnitude as the Fermi β-decay coupling strength but with different dimensions. Gravitational interactions are known to be the weakest interactions in nature. It is the latter property of gravitational interactions that deserve some detailed study. All other interactions in nature must be built out of these weak interactions. The results of this paper seem to demonstrate such a possibility.

We used the coupling constant g_E in the form p^2q^{-2} to introduce a constant of the dimension of a length. This procedure has, in a natural way, via general relativity, leads us to a unified structure of symmetric and anti-symmetric fields. The formal results of the theory show clearly that tensor and spinor descriptions of the fields can be integrated into a single geometrical scheme. In particular, integral and half integral spin fields have resulted from the requirement of general covariance and from the treatment of symmetric and anti-symmetric fields as a single geometrical entity.

Now, why should a theory constructed purely in a deductive way have any relation to physical reality, to observation? It is not possible to discover

a unified field theory by a clear cut, sharp representation of experimental data since such data will always be incomplete and insufficient. A unified field theory is either a grand synthesis of present and future experimental data or just a fruitless intellectual effort. We can only be guided by some general principles, by historical and present trends and above all by what we believe to understand and to cherish most.

RIASSUNTO (*)

Si dimostra che la teoria generalizzata della gravitazione di Einstein nella modificazione dell'Autore contiene campi di spin integrali e semi-integrali. Una costante avente le dimensioni di una lunghezza ha un ruolo fondamentale nella distinzione fra campi di tipo bosonico e di tipo fermionico.

(*) *Traduzione a cura della Redazione.*

Some Predictions from a Composite Model of Baryons (*).

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(ricevuto il 12 Novembre 1959)

Summary. — Baryons are considered as composites of a neutral baryon core plus various mesons. This model makes three specific predictions: 1) The K^+-$\Sigma^0(\Lambda^0)$ force should be much more attractive than the K^--$\Sigma^0(\Lambda^0)$ force. 2) Neutral particles should be approximately electrically neutral throughout. 3) At high energies many elementary particle reactions should have characteristic angular distributions.

Fig. 1 is a three-dimensional representation, drawn in perspective, of the masses of the elementary particles other than leptons and the heavy antiparticles. Mass is plotted in the up direction, strangeness to the right, and electric charge out from the plane of the paper.

Fig. 2 is the same as Fig. 1, but symmetrized in appearance by the following changes: The masses of all baryons are contracted to one value, and likewise for the mesons. Also the baryons are moved one unit of strangeness to the right. The pattern of the baryons in the charge-strangeness plane is then just the same as that of the mesons except for the lack of another π^0-meson analogous to the Λ^0. Since conservation of baryons insures that only differences in strangeness between baryons are observable, there will be no change in the strangeness selection rules. This transformation has been described by Schwinger [1] as part of a general theory of elementary particles.

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(**) Alfred P. Sloan Research Fellow.

[1] J. Schwinger: *Ann. of Phys.*, **2**, 407 (1957).

As has been pointed out by DALLAPORTA *et al.* (2), the resultant congruency in charge-strangeness space of the pattern of the baryons and the pattern of the strongly interacting mesons suggests that every baryon is made up

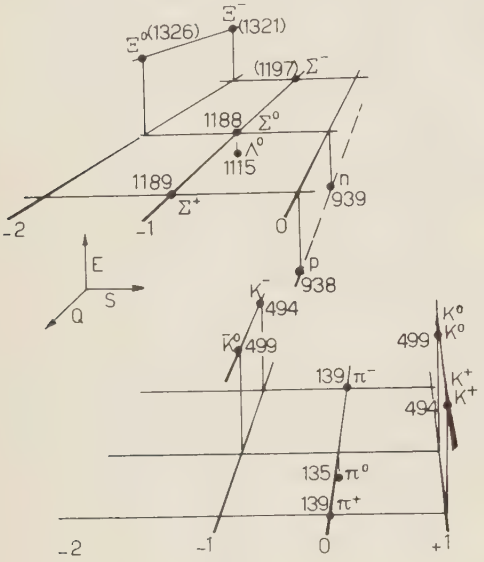

Fig. 1.

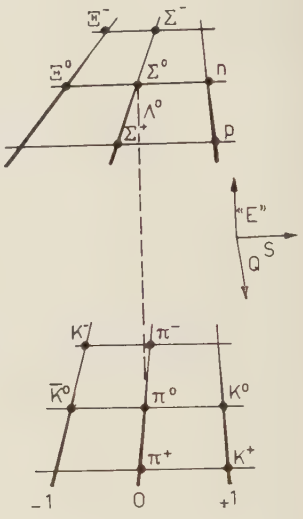

Fig. 2.

out of a heavy neutral baryon core (B_0) plus mesons. That is, Σ's and Λ^0 are each a composite of a universal neutral spin $\frac{1}{2}$ core B_0 plus π-mesons and pairs of K-mesons; nucleons and Ξ's each contain, in addition, a single K-meson. The conservation of strangeness is a result of the conservation of the constituent B_0's and of the evenness or oddness of the total number of K's in the elementary particles. In the present discussion we will assume also that incident π-mesons can be absorbed by single K-mesons inside baryons.

Such a composite model has well-known defects, but at least makes three predictions. (Since the heavy antiparticles are similarly thought of as composed of the charge conjugates of both the baryon core and the mesons, similar predictions are made for antiparticles also.)

The predictions are:

1) The large $\Xi^- - \Lambda^0 - p$ mass difference indicates a great difference between the $K^-(K^0) \vdash B_0$, and $K^-(\bar{K}^0) \dashv B_0$ interactions. In particular, at low energies the force between K^+ and Σ^0 or Λ^0 should be much more attractive than that between K^- and Σ^0 or Λ^0. This strong asymmetry in interactions is what the conventional asymmetry in strangeness is traded for.

(2) *Proc. CERN Conference* (1958), p. 167.

2) The electric charge distribution of every baryon should have a neutral core, and neutral particles should be approximately electrically neutral throughout.

3) Associated production reactions and charge-exchange scattering of K's at energies above the pure S-wave region should show angular distributions characteristic of pick-up processes as follows (the favored direction of emission of each particle in the C.M. system is shown by an arrow):

a) $\pi^+ + \mathrm{p} \to \vec{\Sigma}^+ + \overleftarrow{\mathrm{K}}^+$

b) $\pi^+ + \mathrm{n} \to \vec{\Sigma}^+ + \overleftarrow{\mathrm{K}}^0$

c) $\pi^+ + \mathrm{n} \to \vec{\Sigma}^0 (\vec{\Lambda}^0) + \overleftarrow{\mathrm{K}}^+$

d)* $\pi^- + \mathrm{p} \to \overrightarrow{\Sigma}^- + \overleftarrow{\mathrm{K}}^+$

e)* $\pi^- + \mathrm{p} \to \overleftarrow{\Sigma}^0 (\overleftarrow{\Lambda}^0) + \vec{\mathrm{K}}^0$

f) $\mathrm{K}^+ + \mathrm{n} \to \vec{\mathrm{p}} + \overleftarrow{\mathrm{K}}^0$

g) $\mathrm{K}^0 + \mathrm{p} \to \vec{\mathrm{n}} + \overleftarrow{\mathrm{K}}^+$

h) $\overline{\mathrm{K}}^0 + \mathrm{p} \to \vec{\Xi}^0 + \overleftarrow{\mathrm{K}}^+$

i) $\overline{\mathrm{K}}^0 + \mathrm{n} \to \vec{\Xi}^0 + \overleftarrow{\mathrm{K}}^0$

j) $\mathrm{K}^- + \mathrm{p} \to \overrightarrow{\Xi}^- + \overleftarrow{\mathrm{K}}^+$

k) $\mathrm{K}^- + \mathrm{n} \to \overrightarrow{\Xi}^0 + \overleftarrow{\mathrm{K}}^0$

Prediction 1) cannot be checked at present, since the K-$\Sigma(\Lambda)$ forces are as yet unknown. 2) agrees well with the measured [3] charge distribution of the neutron. 3) agrees qualitatively (marked *) with what data [4] exists on reactions *d*) and *e*); the data on reaction *a*) are under dispute [5].

[3] E. Melkonian, B. M. Rustad and W. W. Havens jr.: *Phys. Rev.*, **114**, 1571 (1959).

[4] *Proc. CERN Conference* (1958), pp. 147-151.

[5] F. S. Crawford jr., R. L. Douglass, M. L. Good, G. R. Kalbfleisch, M. L. Stevenson and H. K. Ticho: *Phys. Rev. Lett.*, **3**, 394 (1959).

RIASSUNTO (-)

I barioni si considerano come aggregazioni di nuclei neutri di barioni più vari mesoni. Questo modello genera tre previsioni specifiche: 1) la forza K^+-$\Sigma^0(\Lambda^0)$ deve essere più attrattiva della forza K^--$\Sigma^0(\Lambda^0)$. 2) Le particelle neutre devono essere approssimativamente elettricamente neutre dappertutto. 3) Ad alte energie molte reazioni di particelle elementari devono avere distribuzioni angolari caratteristiche.

(*) *Traduzione a cura della Redazione.*

Résultats théoriques et expérimentaux concernant les guides d'ondes chargés et les cavités partiellement coaxiales.

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(ricevuto il 30 Novembre 1959)

Summary. — The authors give a method to resolve systems of equations governing the problem of the dispersion relation in a disk loaded circular waveguide and the problem of the eigenvalue frequency in hybrid cavities used as klystron's resonators and wavemeters. By Fourier analysing field's components till order six, they obtain a very good agreement between theoretical and experimental results.

1. – Introduction.

La solution rigoureuse des équations de Maxwell pour un guide d'ondes chargé par des iris circulaires a été étudiée par W. Walkinshaw [1-3] et par Grosjean [4]. Cependant peu d'applications numériques avec comparaison aux résultats expérimentaux ont été faites, en raison de la longueur des calculs.

Il en est de même, d'une façon générale, de tous les problèmes où l'on se propose de résoudre les équations des champs physiques (électromagnétisme, acoustique, élasticité, etc.) avec valeurs aux limites imposées, dans un domaine géométrique de forme complexe, mais se subdivisant en plusieurs domaines élémentaires de forme simple où les solutions s'obtiennent aisément par la

(1) W. Walkinshaw: *Journ. Appl. Phys.*, **20**, 634 (1949).
(2) W. Walkinshaw and J. S. Bell: AERE Rep. G/R 675.
(3) W. Walkinshaw and J. S. Bell: AERE Rep. T/R 864.
(4) C. C. Grosjean: *Nuovo Cimento*, **1**, 427 (1955).

méthode de la séparation des variables. En effet, on est alors conduit à raccorder, sur des surfaces purement géométriques (non matérielles), des champs représentés par des séries infinies, et on aboutit à un système d'une infinité d'équations linéaires à une infinité d'inconnues: il est évidemment difficile de pousser les calculs jusqu'aux solutions pratiques.

Au cours des études préparatoires à la réalisation d'accélérateurs linéaires d'électrons, nous avions constamment ([5,6]) relevé un certain désaccord entre les résultats des mesures et ceux de la théorie élémentaire ([7]). Aussi, avec l'aide d'une machine à calculer électronique (ordinateur IBM 650), avons-nous cherché à parfaire l'accord entre la théorie et l'expérience. On trouvera ci-dessous le développement des calculs et leur application d'une part aux guides à iris, et d'autre part aux cavités cylindriques hybrides (partiellement coaxiales) utilisées comme ondemètres ou comme résonateurs de klystrons.

2. – Théorie du guide d'ondes chargé par des iris circulaires.

2·1. *Equations générales.* – Dans le canal central ($0 < r < a$), les champs correspondant au mode TM symétrique de révolution s'écrivent en unités MKS rationalisées, au facteur $\exp[i\omega t]$ près:

$$(1)\quad \begin{cases} E_z = \sum_{m=-\infty}^{+\infty} A_m J_0(\mu_m r) \exp[-i\beta_m z], \\ E_r = \sum_{m=-\infty}^{+\infty} \frac{i\beta_m}{\mu_m} A_m J_1(\mu_m r) \exp[-i\beta_m z], \\ Z_0 H_\varphi = \sum_{m=-\infty}^{+\infty} \frac{ik}{\mu_m} A_m J_1(\mu_m r) \exp[-i\beta_m z], \end{cases}$$

avec

$$\mu_m^2 + \beta_m^2 = k^2 = [2\pi/\lambda_0]^2$$

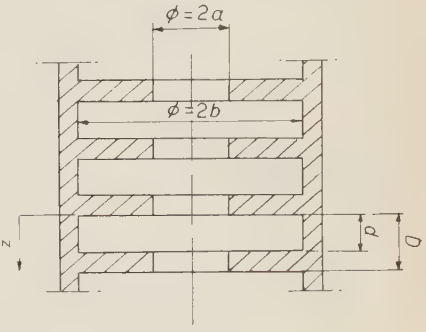

Fig. 1.

(λ_0 est la longueur d'onde dans le vide; Z_0 est l'impédance du vide 376.73 Ω). Il s'agit d'ondes progressives, le sens de propagation dépendant du signe de β_m. Les constantes de phase sont comprises dans les A_m, quantités complexes. On sait ([7]) qu'on est conduit à poser, en appliquant le théorème de Floquet-

([5]) R. Combe et M. Feix: Rapport CNET no. 1169.
([6]) R. Combe: *Thèse* (Paris, 1956).
([7]) W. Walkinshaw: *Proc. Phys. Soc.*, **61**, 246 (1948).

Bloch (λ_g longueur d'onde guidée)

$$\beta_m = \beta_0 + m\frac{2\pi}{D} = 2\pi\left(\frac{1}{\lambda_g} + \frac{m}{D}\right).$$

Dans la zone extérieure, pour une corrugation déterminée, les champs s'écrivent

$$(2)\qquad \begin{cases} E_z = \sum_{p=0}^{+\infty} B_p F_0(\chi_p r)\cos\left(p\dfrac{\pi z}{d}\right), \\ E_r = \sum_{p=0}^{+\infty} p\dfrac{\pi}{d}\dfrac{1}{\chi_p} B_p F_1(\chi_p r)\sin\left(p\dfrac{\pi z}{d}\right), \\ Z_0 H_\varphi = \sum_{p=0}^{+\infty} \dfrac{ik}{\chi_p} B_p F_1(\chi_p r)\cos p\left(\dfrac{\pi z}{d}\right), \end{cases}$$

avec

$$\chi_p^2 + p^2\left(\frac{\pi}{d}\right)^2 = k^2.$$

F_0 et F_1 sont définis par

$$\begin{cases} Y_0(\chi_p b)\, F_0(\chi_p r) = J_0(\chi_p r)\, Y_0(\chi_p b) - J_0(\chi_p b)\, Y_0(\chi_p r), \\ Y_0(\chi_p b)\, F_1(\chi_p r) = J_1(\chi_p r)\, Y_0(\chi_i b) - J_0(\chi_i b)\, Y_1(\chi_i r), \end{cases}$$

Il s'agit maintenant d'ondes stationnaires. Les constantes de phase sont comprises dans les B_p, quantités complexes.

2·2. *Conditions aux limites.* – Elles sont vérifiées dans les corrugations, le métal étant supposé être un conducteur parfait:

$r = b$, $0 < z < d$ donne $E_z = 0$ et $a < r < b$, $z = 0$, donne $E_r = 0$.

Nous allons de plus écrire que pour $r = a$, $0 < z < D$, les champs E_z et H_φ, calculés dans le canal central et dans la corrugation, sont égaux. La condition d'égalité sera alors aussi vérifiée par E_r, d'après les équations de Maxwell.

2·3. *Champ électrique.* – Posons

$$f(z) = E_z(r = a).$$

Dans la zone intérieure on a, d'après (1)

$$f(z) = \sum_{m=-\infty}^{+\infty} A'_m \exp\left[-i\beta_m z\right] \tag{3}$$

avec

$$A'_m = A_m J_0(\mu_m a)\,. \tag{4}$$

Dans la corrugation, on a, pour $0 < z < d$, d'après (2)

$$f(z) = \sum_{p=0}^{+\infty} B'_p \cos\left(p\,\frac{\pi z}{d}\right), \tag{5}$$

avec

$$B'_p = B_p F_0(\chi_p a)\,. \tag{6}$$

D'autre part, au contact du métal, pour $d < z < D$, on a simplement

$$f(z) = 0\,. \tag{7}$$

Appliquos à la formule (3) le calcul classique des coefficients d'une série de Fourier, $f(z)$ étant défini par (5) et (7). On trouve

$$\left|\begin{array}{l} A'_m = \dfrac{1}{D}\displaystyle\int_0^D f(z)\exp\left[i\beta_m z\right]\mathrm{d}z = \dfrac{1}{D}\int_0^d \left[\sum_{p=0}^{\infty} B'_p \cos\left(p\,\dfrac{\pi z}{d}\right)\right]\exp\left[i\beta_m z\right]\mathrm{d}z\,, \\ A'_m \quad \displaystyle\sum_{p=0}^{\infty} R_{mp} B'_p \end{array}\right. \tag{8}$$

avec

$$R_{mp} = \frac{i\beta_m}{D}\,\frac{1-(-1)^p \exp\left[i\beta_m d\right]}{\beta_m^2 - p(\pi^2/d^2)}\,. \tag{9}$$

2·4. *Champ magnétique.* – Posons

$$g(z) = Z_0 H_\varphi(r = a)\,.$$

Dans la zone intérieure, on a, d'après (1)

$$g(z) = \sum_{m=-\infty}^{+\infty} A''_m \exp\left[-i\beta_m z\right] \tag{10}$$

avec

$$A''_m = \frac{ik}{\mu_m} A_m J_1(\mu_m a)\,. \tag{11}$$

Nous poserons aussi

$$\alpha_m = \frac{A''_m}{A'_m} = \frac{ik}{\mu_m}\,\frac{J_1(\mu_m a)}{J_0(\mu_m a)}\,. \tag{12}$$

Dans la corrugation, on a pour $0 < z < d$, d'après (2)

$$g(z) = \sum_{p=0}^{\infty} B''_p \cos\left(p\frac{\pi z}{d}\right), \tag{13}$$

$$B''_p = \frac{ik}{\chi_p} B_p F_1(\chi_p a) . \tag{14}$$

Nous poserons aussi

$$\gamma_p = \frac{B''_p}{B'_p} = \frac{ik}{\chi_p}\frac{F_1(\chi_p a)}{F_0(\chi_p a)} . \tag{15}$$

La fonction nous intéresse seulement dans l'intervalle $(0, d)$. (En effet le long de la paroi métallique, pour $d < z < D$, le champ H_y satisfait automatiquement la condition aux limites.) Complétons-la dans l'intervalle $(-d, 0)$ par la function $\exp[i\beta_m z]$. Cette fonction complétée, notée $h(z)$, définie dans l'intervalle $(-d, d)$ est une fonction paire et peut être développée en série cosinusoïdale de Fourier

$$h(z) = \sum_{p=0}^{+\infty} S_{mp} \cos\left(p\frac{\pi z}{d}\right) . \tag{16}$$

On trouve

$$S_{mp} = ② \frac{i\beta_m}{d} \frac{1 - (-1)^p \exp[-i\beta_m d]}{p^2(\pi^2/d^2) - \beta_m^2} \tag{17}$$

le coefficient ② devant être remplacé par 1 si $p = 0$.

Nous pouvons écrire d'après (10) et (16)

$$g(z) = \sum_{m=-\infty}^{+\infty} A''_m \sum_{p=0}^{+\infty} S_{mp} \cos\left(p\frac{\pi z}{d}\right) . \tag{18}$$

Identifions les coefficients de $\cos(p(\pi z/d))$ dans (13) et (18), il vient:

$$B''_p = \sum_{m=-\infty}^{+\infty} S_{mp} A''_m$$

ou

$$\gamma_p B'_p = \sum_{m=-\infty}^{+\infty} S_{mp} \alpha_m A'_m . \tag{19}$$

2·5. *Equation de la dispersion.* – Eliminons les A'_m entre (19) et (8), l'équation (8) étant récrite

$$A'_m = \sum_{q=0}^{+\infty} R_{mq} B'_q .$$

Nous obtenons un système d'une infinité d'équations, (les inconnues étant les amplitudes B'), l'équation de rang p s'écrivant

$$\sum_{m=-\infty}^{+\infty} \alpha_m S_{mp} \sum_{q=0}^{+\infty} R_{mq} B'_q - \gamma_p B'_p = 0 . \tag{20}$$

ou, d'une manière condensée:

$$\sum_{q=0}^{+\infty} Z_{pq} B'_q = 0 . \tag{21}$$

avec

$$Z_{pq} = \sum_m \alpha_m S_{mp} R_{mq} - \delta_{pq} \gamma_q . \tag{22}$$

Tous calculs faits, Z_{pq} s'écrit

$$Z_{pq} = \frac{4}{dD} \sum_{m=-\infty}^{+\infty} \frac{\beta_m^2 J_1(\mu_m a)}{\mu_m J_0(\mu_m a)} \cdot \frac{\Xi}{(p^2(\pi^2/d^2) - \beta_m^2)(q^2(\pi^2/d^2) - \beta_m^2)} - \xi \delta_{pq} \frac{1}{X_q} \frac{F_1(X_q a)}{F_0(X_q a)} . \tag{23}$$

avec

$$\Xi = \begin{cases} 1 - \cos \beta_m d & \text{si } p \text{ et } q \text{ sont pairs} \\ i \sin \beta_m d & \text{si } p \text{ est pair et } q \text{ impair} \\ -i \sin \beta_m d & \text{si } p \text{ est impair et } q \text{ pair} \\ 1 + \cos \beta_m d & \text{si } p \text{ et } q \text{ sont impairs} \end{cases} \quad \text{et} \quad \xi = \begin{cases} 2 \text{ si } p = q = 0 \\ 1 \text{ dans tous les autres cas} \end{cases}$$

Le système d'équations (21) n'admet de solutions non identiquement nulles que si le déterminant des coefficients est nul

$$|Z_{pq}| = 0 . \tag{24}$$

Cette équation (24) est l'équation de la dispersion, qui relie β_m (c'est-à-dire β_0 ou la longueur d'onde guidée λ_g, ou la vitesse de propagation $v = f\lambda_g$) à k (c'est-à-dire à la longueur d'onde dans le vide λ_0, ou à la fréquence $f = c/\lambda_0$), et aux paramètres géométriques a, b, d, D.

Les termes Z_{pq} sont alternativement réels ou imaginaires purs. Le déterminant est hermitique. Son calcul par la méthode des pivots [8] ne présente pas de difficultés spéciales. Pratiquement, on imposait les valeurs de k et

[8] W. E. Milne: *Numerical Calculus* (Princeton, 1949), chap. 1.

des dimensions a, b, d, D. Pour une valeur d'essai β_0, on peut calculer les éléments Z_{pq}, et la valeur du déterminant lui-même, limité bien entendu dans l'approximation d'ordre p à $p+1$ lignes et $p+1$ colonnes (chaque terme est obtenu par une sommation sur m, m variant ordinairement de $-p$ à $+p$).

Lorsqu'on dispose d'une série de valeurs du déterminant pour différentes valeurs de β_0, on calcule, par interpolation, la valeur correcte de β_0: p variait de 3 à 6. Pour $p = 6$, la durée du calcul d'une valeur du déterminant, effectué avec un ordinateur IBM 650 est d'environ 5 minutes.

3. – Résultats concernant les guides d'ondes chargés.

Nous avons calculé la courbe de dispersion des guides nos. 4 et 5 précédemment décrits ([5,6]), dont nous rappelons les caractéristiques:

	Guide no. 4	Guide no. 5
$2a$ (mm)	23.36	46.81
$2b$ (mm)	79.12	87.84
d (mm)	7.50	7.50
D (mm)	10.00	10.00
Valeur de E_0 (V/cm)	36000	9000

E_0 est l'amplitude du champ électrique E_z lorsque la puissance transmise calculée en intégrant le flux du vecteur de Poynting correspondant au terme fondamental ($m = 0$) est égale à 500 kW.

Ces deux guides ont d'abord été calculés, par la théorie élémentaire, pour qu'à la fréquence 3000 MHz ($\lambda_0 = 10.00$ cm) la longueur d'onde guidée soit $\lambda_g = 10.00$ cm (point A de la Fig. 2) c'est-à-dire pour que la vitesse de phase soit égale à c. Or l'écart entre les résultats des mesures et ce calcul simple est très notable, notamment (pour le guide n. 4) puisqu'on a trouvé expérimentalement:

guide no. 4: $\lambda_0 = 10.00$ cm $\lambda_g = 15.20$ cm,

guide no. 5: $\lambda_0 = 10.00$ cm $\lambda_g = 11.10$ cm.

La Fig. 2 rassemble les courbes théoriques de dispersion, calculées par la théorie rigoureuse, avec en surcharge les points expérimentaux. On constate que l'accord est très satisfaisant.

Les mesures ont été faites d'une part par la méthode déjà décrite ([5,6]): on relève à l'aide d'une sonde magnétique les diagrammes d'ondes stationnaires,

et on mesure ainsi λ_g en fonction de λ_0; d'autre part par la méthode de la résonance: on ferme le guide à ses deux extrémités par des courts-circuits et on cherche les fréquences de résonance de la cavité ainsi constituée.

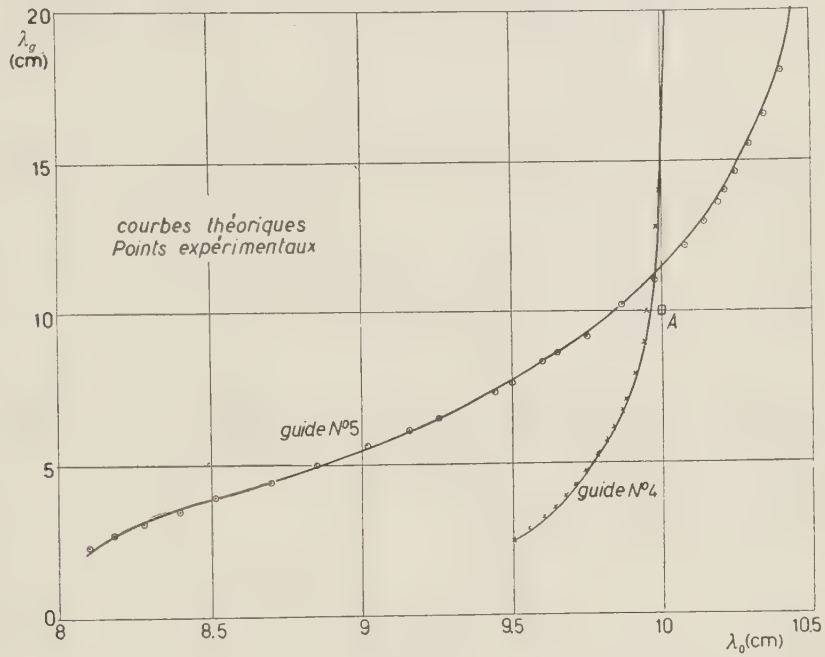

Fig. 2.

4. - Théorie de la cavité cylindrique partiellement coaxiale (ondemètres, klystrons). Mode TM de révolution.

Dans la zone ①, les champs correspondants s'écrivent:

$$(25)\quad \begin{cases} E_z = \sum_{m=0}^{+\infty} A_m J_0(\mu_m r) \cos\left(m\dfrac{\pi z}{d}\right), \\ E_r = \sum_{m=0}^{+\infty} m\dfrac{\pi}{d}\dfrac{1}{\mu_m} A_m J_1(\mu_m r) \sin\left(m\dfrac{\pi z}{d}\right), \\ Z_0 H_\varphi = \sum_{m=0}^{+\infty} \dfrac{ik}{\mu_m} A_m J_1(\mu_m r) \cos\left(m\dfrac{\pi z}{d}\right), \end{cases}$$

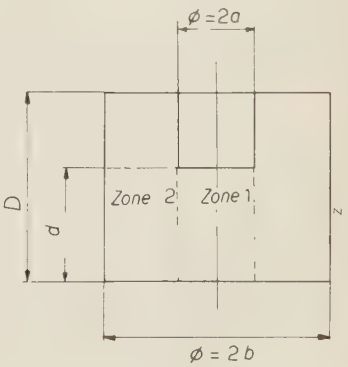

Fig. 3.

et dans la zone ②:

$$(26)\qquad \begin{cases} E_z = \sum_{p=0}^{+\infty} B_p F_0(\chi_p r)\cos\left(p\dfrac{\pi z}{D}\right), \\ E_r = \sum_{p=0}^{+\infty} p\dfrac{\pi}{D}\dfrac{1}{\chi_p} B_p F_1(\chi_p r)\sin\left(p\dfrac{\pi z}{D}\right), \\ Z_0 H_\varphi = \sum_{p=0}^{+\infty} \dfrac{ik}{\chi_p} B_p F_1(\chi_p r)\cos\left(p\dfrac{\pi z}{D}\right), \end{cases}$$

avec les relations habituelles

$$\begin{cases} \mu_m^2 + m^2\dfrac{\pi^2}{d^2} = k^2, \\ \chi_p^2 + p^2\dfrac{\pi^2}{D^2} = k^2. \end{cases}$$

Les conditions aux limites sont satisfaites sur les faces planes et sur la surface cylindrique $r = b$. Il faut raccorder les deux solutions pour $r = a$.

4·1. *Champ électrique.* – Posons

$$f(z) = E_z \qquad (r = a).$$

Dans la zone extérieure ②, on a, d'après (26)

$$(27)\qquad f(z) = \sum_{p=0}^{\infty} B'_p \cos\left(p\frac{\pi z}{D}\right),$$

avec

$$(28)\qquad B'_p = B_p F_0(\chi_p a).$$

Dans la zone intérieure ① on a, pour $0 < z < d$, d'après (25)

$$(29)\qquad f(z) = \sum_{m=0}^{+\infty} A'_m \cos\left(m\frac{\pi z}{d}\right),$$

$$(30)\qquad A'_m = A_m J_0(\mu_m a).$$

D'autre part, au contact du métal, pour $d < z < D$, on a simplement

$$(31)\qquad f(z) = 0.$$

Appliquons à la formule (27) le calcul des coefficients de Fourier, $f(z)$ étant défini par (29) et (31). On trouve

$$(32)\qquad \begin{cases} B'_p = \dfrac{②}{D}\displaystyle\int_0^D f(z)\cos\left(p\frac{\pi z}{D}\right)\mathrm{d}z = \dfrac{②}{D}\displaystyle\int_0^d \left|\sum_{m=0}^{+\infty} A'_m \cos\left(m\frac{\pi z}{d}\right)\right| \cos\left(p\frac{\pi z}{D}\right)\mathrm{d}z\,, \\ B'_p = \displaystyle\sum_{m=0}^{+\infty} R_{pm} A'_m\,, \end{cases}$$

le coefficient ② devant être remplacé par 1 si $p = 0$, avec:

$$(33)\qquad \begin{cases} R_{pm} = 2\dfrac{(-1)^{m+1}}{\pi}\dfrac{p}{D^2}\dfrac{\sin\left(p\dfrac{\pi d}{D}\right)}{\dfrac{m^2}{d^2} - \dfrac{p^2}{D^2}} & p \text{ et } m \neq 0 \\ R_{p0} = \dfrac{2}{\pi p}\sin\left(p\dfrac{\pi d}{D}\right) & R_{0m} = \dfrac{d}{D}\delta_{0m}\,. \end{cases}$$

4·2. *Champ magnétique.* – Posons

$$g(z) = Z_0 H_\varphi \qquad (r = a)\,.$$

Dans la zone extérieure ② on a, d'après (26)

$$(34)\qquad g(z) = \sum_{p=0}^{+\infty} B''_p \cos\left(p\frac{\pi z}{D}\right),$$

$$(35)\qquad B''_p = \frac{ik}{\chi_p} B_p F_1(\chi_p a)\,.$$

Nous poserons aussi

$$(36)\qquad \gamma_p = \frac{B''_p}{B'_p} = \frac{ik}{\chi_p}\frac{F_1(\chi_p a)}{F_0(\chi_p a)}\,.$$

Dans la zone intérieure ①, pour $0 < x < d$, on a, d'après (25)

$$(37)\qquad g(z) = \sum_{m=0}^{+\infty} A''_m \cos\left(m\frac{\pi z}{d}\right),$$

avec

$$(38)\qquad A''_m = \frac{ik}{\mu_m} A_m J_1(\mu_m a)\,.$$

Nous poserons aussi

$$\alpha_m = \frac{A''_m}{A'_m} = \frac{ik}{\mu_m}\frac{J_1(\mu_m a)}{J_0(\mu_m a)}. \tag{39}$$

La fonction $\cos(p(\pi z/D))$ nous intéresse seulement dans l'intervalle $(0, d)$. Développons-la en série cosinusoïdale de Fourier dans l'intervalle $(0, d)$

$$\cos\left(p\frac{\pi z}{D}\right) = \sum_{m=0}^{+\infty} S_{pm}\cos\left(m\frac{\pi z}{d}\right), \tag{40}$$

on trouve

$$\begin{cases} S_{pm} = 2\dfrac{(-1)^{m+1}}{\pi}\dfrac{p}{dD}\dfrac{\sin\left(p\dfrac{\pi d}{D}\right)}{\dfrac{m^2}{d^2}-\dfrac{p^2}{D^2}} & p \text{ et } m \neq 0 \\ S_{p0} = \dfrac{1}{\pi}\dfrac{D}{pd}\sin\left(p\dfrac{\pi d}{D}\right) \\ S_{0m} = \delta_{0m}. \end{cases} \tag{41}$$

Nou pouvons écrire, d'après (34) et (40)

$$g(z) = \sum_{p=0}^{+\infty} B''_p \sum_{m=0}^{+\infty} S_{pm}\cos\left(m\frac{\pi z}{d}\right). \tag{42}$$

Identifions les coefficients de $\cos(m(\pi z/d))$ dans (37) et (42):

$$A''_m = \sum_{p=0}^{+\infty} S_{pm} B''_p$$

ou

$$\alpha_m A'_m = \sum_{p=0}^{+\infty} S_{pm}\gamma_p B'_p. \tag{43}$$

4·3. *Equation aux fréquences propres.* – Cherchons encore à éliminer les A'_m. Changeons p en q dans (43)

$$\alpha_m A'_m = \sum_{q=0}^{+\infty} S_{qm}\gamma_q B'_q.$$

Portons cette expression dans (32), il vient

$$B'_p = \sum_m R_{pm}\frac{1}{\alpha_m}\sum_q S_{qm}\gamma_q B'_q,$$

ou encore, en intervertissant les sommations par rapport aux indices m et q

$$B'_p = \sum_q \gamma_q B'_q \sum_m \frac{1}{\alpha_m} R_{pm} S_{qm} ,$$

et enfin

$$\sum_q Z_{pq} B'_q = 0 \tag{44}$$

avec

$$Z_{pq} = \gamma_q \sum_m \frac{1}{\alpha_m} R_{pm} S_{qm} - \delta_{pq} . \tag{45}$$

Le système d'équations (44) correspond aux équations (21) du premier problème. Tous calculs faits, Z_{pq} s'écrit, en multipliant (*)

si $p \neq 0$, la $p+1^{\text{ème}}$ ligne par $p\pi/\sin\left(p\frac{\pi d}{D}\right)$,

si $p = 0$, la $1^{\text{ère}}$ ligne par 1,

si $q \neq 0$, la $q+1^{\text{ème}}$ colonne par $\frac{q\pi}{\sin(q(\pi d/D))} \frac{\chi_q F_0(\chi_q a)}{F_1(\chi_q a)}$

si $q = 0$, la $1^{\text{ère}}$ colonne par $\frac{1}{2} \frac{\chi_0 F_0(\chi_0 a)}{F_1(\chi_0 a)}$,

$$Z_{pq} = 2\frac{D}{d}\frac{\mu_0 J_0(\mu_0 a)}{J_1(\mu_0 a)} + \sum_{m=1}^{+\infty} 4\frac{D}{d}\frac{\mu_m J_0(\mu_m a)}{J_1(\mu_m a)} \frac{1}{\left(\frac{m^2}{p^2}\frac{D^2}{d^2} - 1\right)\left(\frac{m^2}{q^2}\frac{D^2}{d^2} - 1\right)} \cdots - \tag{46}$$

$$- \delta_{pq} \left(\frac{p\pi}{\sin p \frac{\pi d}{D}}\right)^2 \frac{\chi_p F_0(\chi_p a)}{F_1(\chi_p a)} \qquad (p \neq 0,\ q \neq 0),$$

$$Z_{p0} = Z_{0q} = \frac{\mu_0 J_0(\mu_0 a)}{J_1(\mu_0 a)} \qquad (p \text{ ou } q \neq 0), \tag{47}$$

$$Z_{00} = \frac{1}{2}\left[\frac{d}{D}\frac{\mu_0 J_0(\mu_0 a)}{J_1(\mu_0 a)} - \frac{\chi_0 F_0(\chi_0 a)}{F_1(\chi_0 a)}\right] . \tag{48}$$

Le système d'équations (44) n'admet de solutions non identiquement nulles que si le déterminant des coefficients est nul. Cette condition donne l'équation

(*) Ces transformations ne modifient pas l'équation aux fréquences propres (annulation du déterminant) et simplifient les calculs.

aux fréquences propres qui relie k (c'est-à-dire la longueur d'onde dans le vide λ_0, où la fréquence $f = c/\lambda_0$) aux paramètres géométriques a, b, d, D.

Ce déterminant est réel et symétrique. Nous l'avons également calculé par la méthode des pivots. On imposait les valeurs des dimensions a, b, d, D, et on calculait le déterminant pour un certain nombre de valeurs de k. Une interpolation donnait alors la valeur correcte de k.

5. – Résultats concernant une cavité particulière.

Nous avons appliqué la méthode précédente à la cavité d'un ondemètre à large bande construit dans notre laboratoire, dont les dimensions sont:

$$2a = 10.00 \text{ mm}$$

$$2b = 46.00 \text{ mm}$$

$$D = 41.00 \text{ mm}\,.$$

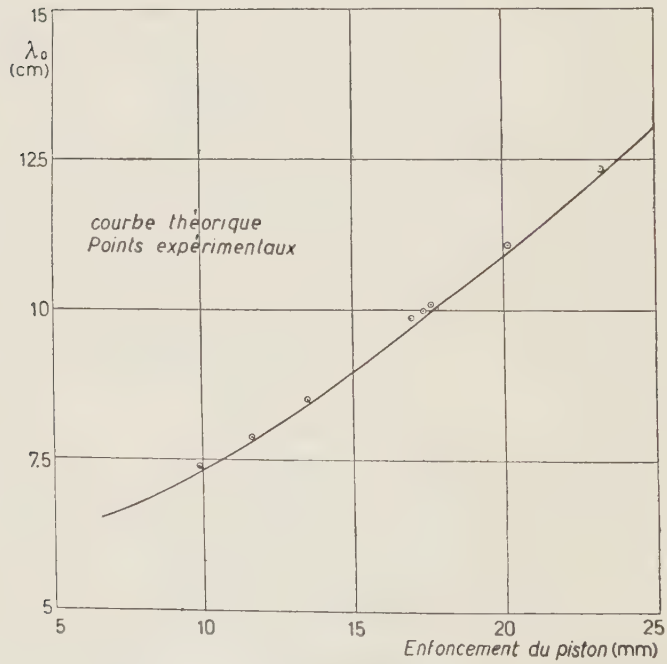

Fig. 4.

La Fig. 4 représente la longueur d'onde de résonance λ_0 en fonction de l'enfoncement $D - d$ du piston. On constate le bon accord entre la théorie et les

mesures. La courbe d'étalonnage expérimental de l'ondemètre a été tracée en utilisant des étalons à quartz, par le Laboratoire National de Radioélectricité.

* * *

Nous exprimons nos vifs remerciements à M. G. Bouquet et à M.elle J. Julien, qui ont programmé les calculs, et à MM. A. Delaunay et G. Michot qui ont construit l'ondemètre et effectué les mesures.

RIASSUNTO (*)

Gli autori espongono un metodo per risolvere i sistemi di equazioni che regolano il problema del rapporto di dispersione in una guida d'onda circolare caricata a dischi ed il problema della frequenza di autovalori nelle cavità ibride usate come risonatori nei klystron e nei misuratori d'onda. Analizzando col metodo di Fourier le componenti del campo sino al sesto ordine ottengono un buon accordo fra i risultati teorici e quelli sperimentali.

(*) *Traduzione a cura della Redazione.*

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Experimental behaviour of Ionic Structures in Liquid Helium - II.

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Summary. — Experiments have been performed to select a proper model among some proposed structures for helium ions in liquid helium. The apparatus is essentially a diode, where the ions are produced by α-rays and the currents can be measured in the bulk liquid, in the liquid-vapour interphase and in the bulk solid. By the quite different behaviour of the positive and negative ions in the interphase experiments one is led to picture the positive ion as a cluster of polarized atoms around one charge, and the negative ion as a large cage where the electron is self-trapping.

1. – Introduction.

The increasing interest in the use of ions in liquid helium as a tool for the investigation of the properties of liquid helium itself, calls attention to the nature of ionic structures. It is the purpose of this paper to discuss the possible structures, and then to describe an experiment which shows some evidence for these model structures for the positive and negative ions.

2. – Possible structures of ions in liquid helium.

In this Section we will briefly review and elaborate on the structures which have been proposed by us (1) and independently calculated by Atkins (2).

2·1. *The positive ion.* – It is well known that in He gas the He_2^+ ion is a stable entity, and elementary calculations indicate that in a dense medium

(1) G. Careri, F. Scaramuzzi and J. O. Thomson: *Nuovo Cimento*, **13**, 186 (1959).
(2) K. R. Atkins: Univ. of Pennsylvania Techn. Rep. no. 3, Contract no. 551 (28).

a cluster He_n^+ must easily be formed, the n atoms sorrounding the positive ion being attached to it by polarization forces. These polarization forces are actually so large that the co-ordinated atoms will stick together at a distance much less than the average atomic distance, behaving like a highly compressed solid droplet. On this ground ATKINS has calculated the mass of a cluster due to this electrostrictive effect to amount to about 50 ^{4}He mass units. We point out here that this mass excess must be considered as a static effect, and has nothing to do with any hydrodynamical effective mass which may exist in specific situations, and may eventually be added to the above indicated static mass. Quite soundly ATKINS also observes that these clusters should be stable as long as the ion moves with a velocity lower than the sound velocity, the sound velocity being that rate at which a compression in the medium can propagate. For larger velocities the charge may leave the cluster and its mobility under an external electric field may then sharply increase, but it is not known if this will be at all the case due to possible creation of excitations by the hot charge cluster.

However we are faced with another possibility. The positive hole may migrate, jumping from one atom to another by random walk, as happens in similar situations in crystalline mediums. The jumping of the positive charge is, of course, in reality the transferring of an electron from a neutral atom to the positive ion, creating an ion from the formerly neutral atom. As ATKINS points out, the electrostrictive effect will soon be obliterated and we will have a quite different entity, a charge distribution in a quite large volume of low density. ATKINS discards this possibility with the argument that this could be the case only if the positive ion were monoatomic. In a cluster, he says, the jumping of a hole is no longer possible since this would involve rearrangement of the massive nuclei, and this cannot occur with a velocity greater than the zero point velocity of the He atoms.

It is our opinion, instead, that the charge may well escape the potential barrier which keeps it in the denser material, by some kind of tunnelling effect. Therefore the question of the possible structures of the positive ion is still an open one, and one has to choose between two limiting models:

I) the positive ion is a solid cluster of He atoms polarized around a positive charge, which may change site while still remaining inside the cluster;

II) the positive ion is a charge distributed in a large region with density comparable to that of the liquid, the charge continuously jumping from one atom to another.

2·2. *The negative ion.* – It is also well known that in He the He^- is unstable, and He_2^- must be loosely bound if it forms at all. It may be that He_n^- has some

stability, and the resulting structure would then be similar to the I) structure proposed above for the positive ion.

However, if the electron likes to stay in the empty space, then the interesting possibility exists that it might remain free in an effective cage, big enough to reduce its zero point motion, and self-trapping inside this cage by polarization forces. It seems that this possibility has been first independently pointed out by FERRELL (3) to explain the anomalous large life of positronium in liquid helium; as a matter of fact, in this way a positronium atom is able to avoid contact with the liquid and increase its lifetime. For this model ATKINS calculated that the extra mass associated with it may be as large as 280 ^{4}He atoms, the negative cloud extending over a region encompassing many helium atoms.

However, here again a tunnelling effect may exist, which could let the electron leave the shell-like cage of polarized atoms, and diffuse in a large region. Therefore we are still faced with the following possibilities:

I) The negative ion is a solid cluster, like the structure I) proposed for the positive ion.

II) The negative ion is a cloud of charge in a cage, self-trapped by the shell of polarized atoms.

III) The negative ion is a free electron moving in a large region, of the density of the liquid, escaping by tunnelling any trapping which might occur.

To the above possibilities, we must add the following one, suggested by WILLIAMS (4) and supported by analogies in the gas:

IV) The negative ion is actually a charged impurity (probably oxygen) which can form a cluster in the same way as the structure I) of the positive ions.

3. – Experimental.

In this Section we will describe an experiment of a qualitative nature, which in our opinion cap help one to choose between the ionic structures above outlined. In this experiment the ions are accelerated by the uniform electric field between the plates of a parallel plate capacity and the current thereby

(3) R. A. FERRELL: *Phys. Rev.*, **108**, 167 (1957).
(4) R. L. WILLIAMS: *Canad. Journ. Phys.*, **35**, 134 (1957).

produced is measured as a function of the applied voltage. The gap between the horizontal plates would be filled with liquid or solid helium, or else partially filled with liquid so that the liquid-gas phase boundary was parallel to and between the plates.

3·1. *The apparatus.* – The apparatus is essentially a diode where the ions are produced by a layer of polonium at one plate, and collected on the other plate by a suitable electric field. Different diodes have been employed and here in the following we give their dimensions in cm (the symbols are shown in Fig. 1) and the relative current density J in units of 10^8 ions/s cm^2, as measured at the standard condition of 2 °K and 200 V/cm.

The plates were made of silver except for the diode No. 1 where platinum was employed. The polonium source was obtained simply by letting one drop of 0.5 normal nitric solution to evaporate on the plate. It is estimated that 10^{11} ion pairs per second were also produced, but the recombination in the liquid was so effective to get current densities ranging only between 10^{-15} to 10^{-10} A/cm^2 s. Incidentally, the range of α-particles in the liquid was lower than 0.3 mm.

The diode was protected sometimes by a plexiglass can, and the whole unit could be moved in the dewar by means of « O » ring seals to follow the bath level when the two phase system was investigated.

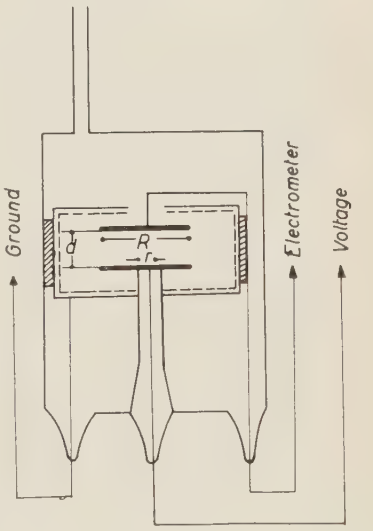

Fig. 1. – The diode used for the liquid and liquid-vapour phases. The helium cryostat is not shown.

The electrometer and other technical devices used here were the same as in the experiments described by Careri, Scaramuzzi and Thomson [1].

	R	r	d	J
no. 1	1	0.25	0.5	2.0
no. 2	1	0.25	0.5	2.5
no. 3	1	0.13	0.8	3.0

A somewhat different diode was used for measuring the current in solid helium, due to the need of applying on high pressure. However, the changes were only matter of technical details.

3·2. *Experimental results*:

A) Classical liquids. As a reference view-point for the behaviour of the diodes we have investigated some non-polar classical liquids. CCl_4 and N_2 were chosen for matter of experimental convenience (Fig. 2 and 3).

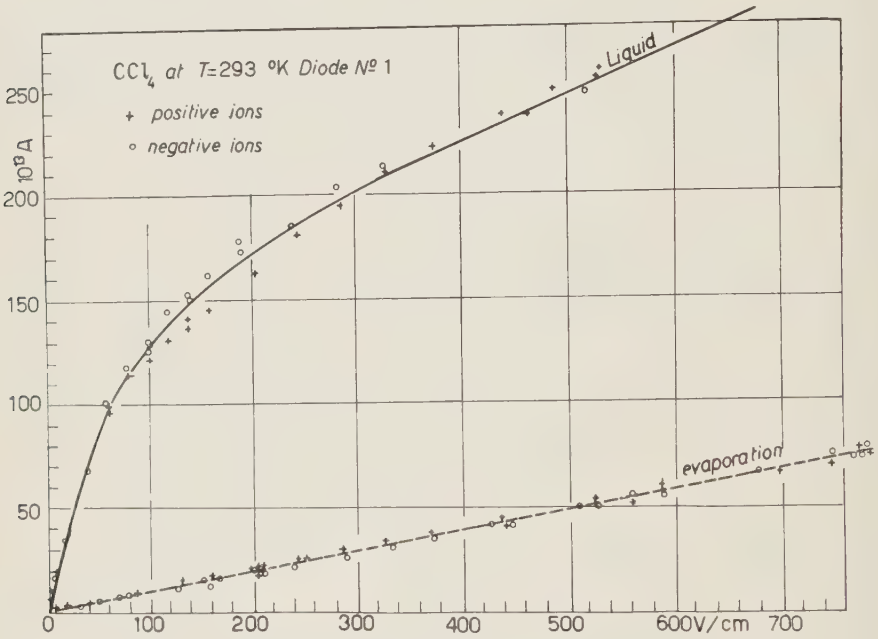

Fig. 2. – The ionic currents in carbon tetrachloride versus the applied field.

The curves labelled « liquid » have been obtained with both the electrodes below the liquid level of the bath, and the curves labelled « evaporation » have been obtained when the upper electrode was above the bath level and the lower one containing the radioactive layer below. The name « evaporation » is a short and improper one for a phenomenon which is a non-equilibrium process of ion extraction from a liquid surface by means of an electric field perpendicular to the surface.

A glance to these curves shows that the behaviour is just the one we should expect, namely:

a) There is no difference between positive and negative ions both in the liquid and in evaporation.

b) By increasing the voltage in the liquid the current increases without approaching a saturation value, in accordance with previous works; this is due to large recombination in the columns.

c) The temperature has a slight influence in the sense of increasing the current with the increasing temperatures as expected in any process involving an activation energy.

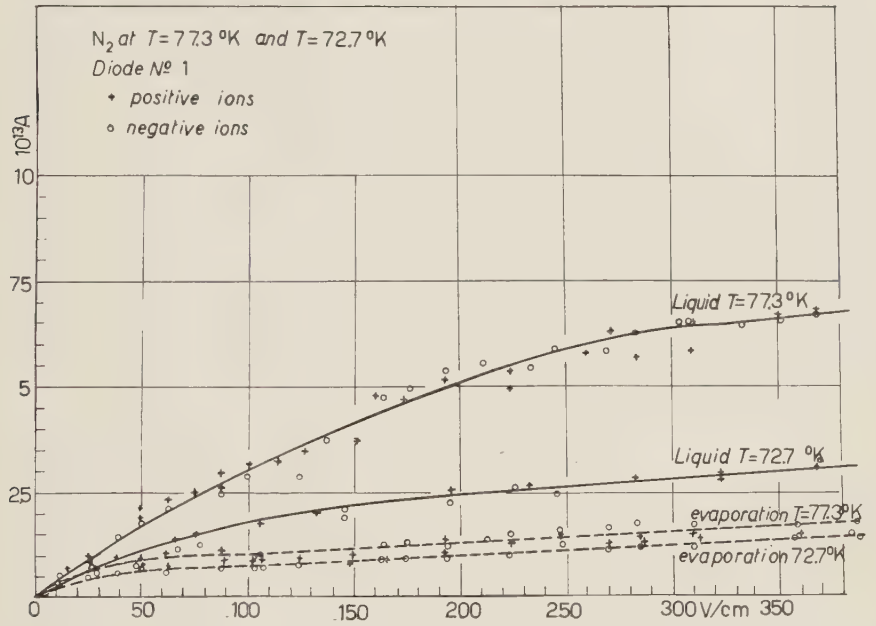

Fig. 3. – The ionic currents in nitrogen versus the applied field at two temperatures.

d) The evaporation currents are lower than the liquid ones, at given voltage and temperature, since the polarization forces keep the ions in the denser medium.

e) Finally, there is not a qualitative difference between CCl_4 and N_2.

B) Liquid helium. The same runs have been performed in liquid helium, and some important differences are at once noticed. The data were well reproducible working with different diodes; only some selected examples are shown in the following.

The evaporation curves also show some important new features, as may be seen in Figs. 4, 5, 6, 7 and 8 that may be summarized as follows:

a) Above T_λ the evaporation curves are the same as the liquid ones and are the same for ions of both signs.

b) Below T_λ the evaporation curve of the positive ions is always lower than the one of the negative ions, the liquid curves remaining the same for both; the difference between evaporation and liquid curves is most drastic for the positive ions just below T_λ.

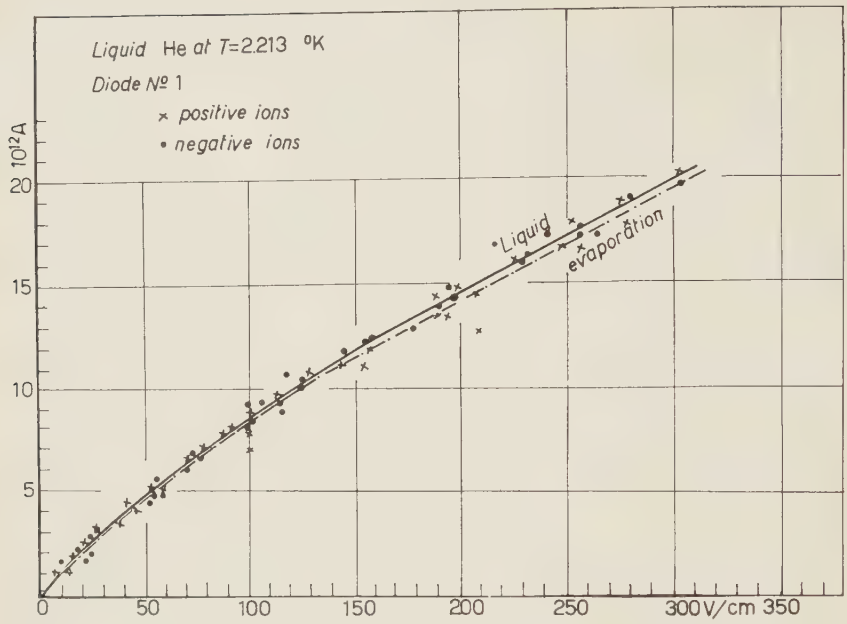

Fig. 4. – The ionic currents in helium versus the applied field, soon above the λ transition.

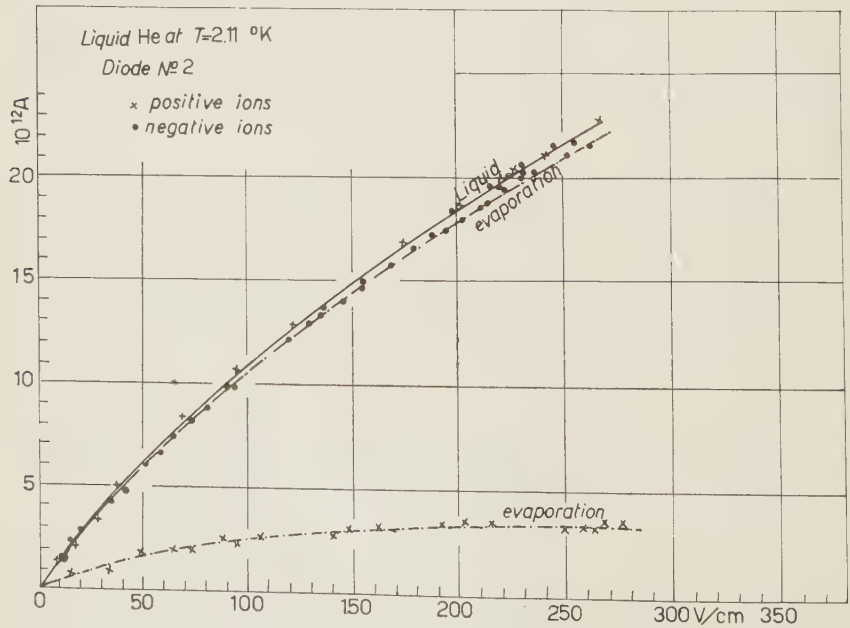

Fig. 5. – The ionic currents in helium versus the applied field, soon below the λ transition. Note the strong effect on the evaporation of the positive ions.

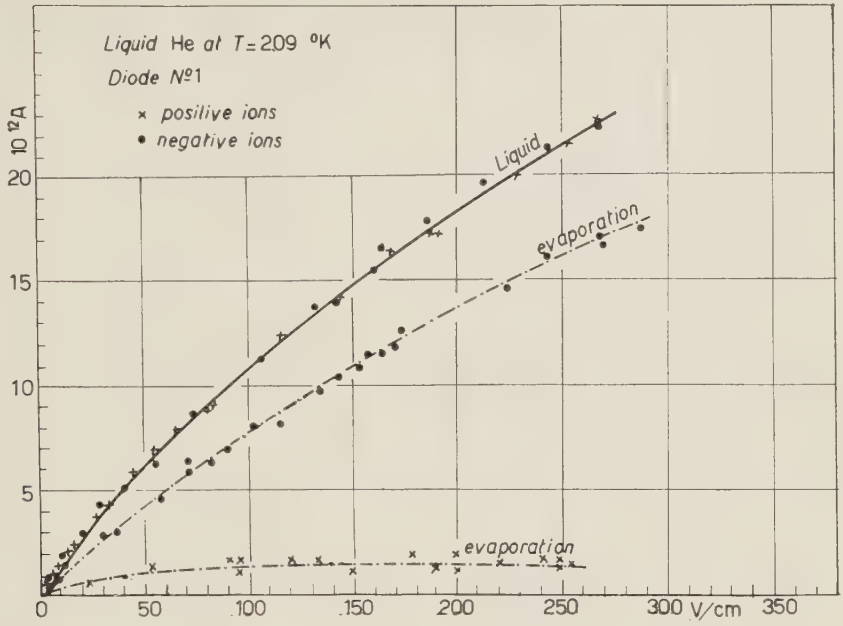

Fig. 6. – The same as Fig. 5, at a lower temperature.

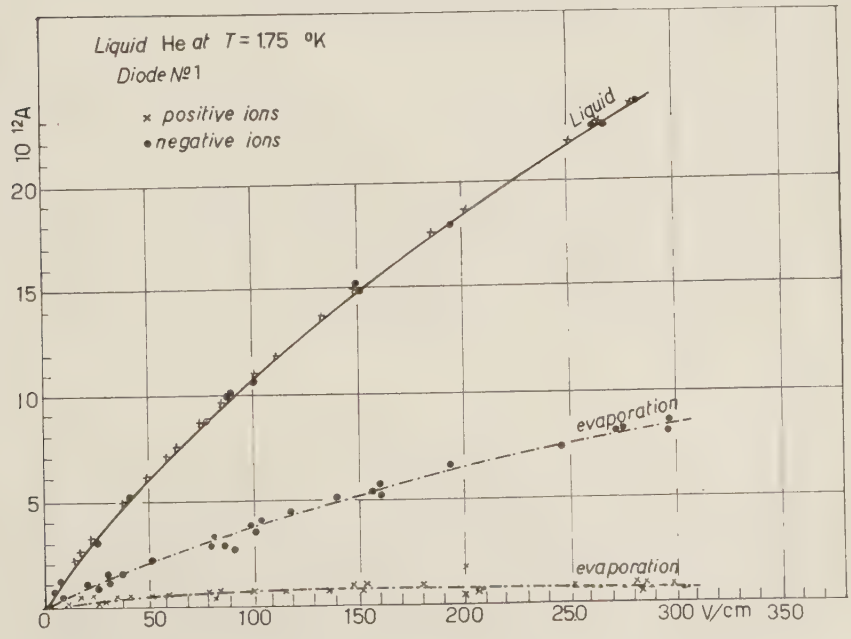

Fig. 7. – The same as Fig. 5, at a lower temperature.

d) At the lowest temperatures the evaporation curves approach very small values which have been proved to be due to leakage current, radioactive contamination of the plexiglas and other similar effects.

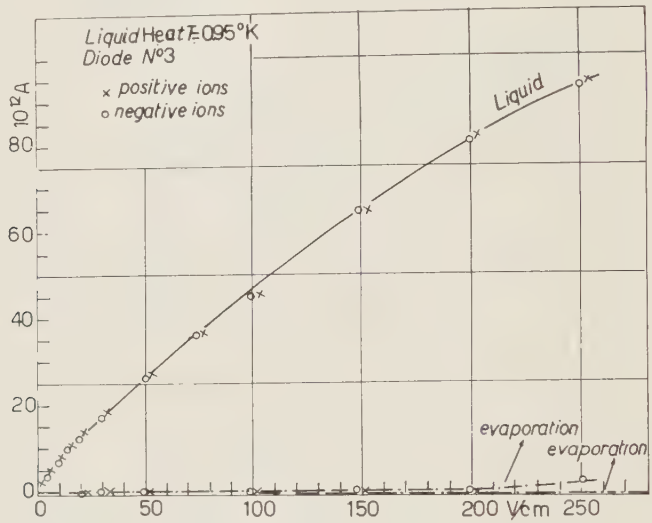

Fig. 8. – The same as Fig. 5, at a lower temperature.

C) Solid helium. When the pressure on the helium in the cell containing the diode was increased to the melting pressure, the current at once fell to an extremely low value, not distinguishable from the electrometer background noise.

Since the experiment necessarily deals with ions of both signs at the same time, we can only say that this result means that at least one of the two species of ions could not move. This would cause the total current to vanish because of the polarization of one electrode. It seems that the negative ion if it is a free electron ought to be mobile in solid helium as in any dielectric. We then interpret the vanishing of the total current as an indication that the current of positive ions is neagligible in the solid.

4. – Discussion.

The above quoted results of the inter-phase experiment may be qualitatively understood in terms of the current picture of liquid helium. The rapid fall of the ionic evaporation below the λ temperature is then due to the rapid fall in the number of excitations, since excitations are needed to let the ions across the liquid-vapour surface. Therefore this extraction mechanism in liquid helium is not much different from the evaporation mechanism of any liquid, where the molecule to evaporate has to pick up some energy from the

sorrounding molecules during a local fluctuation of the collisions, in order to overcome the binding energy with the neighbour molecules. In the case of the ions we presume this binding energy to be much larger due to the strong forces ion-polarized molecules, and the ions will have definite preference to stay inside the liquid medium.

Once this picture is accepted, then we deduce that the positive and negative ions must be quite different entities, and that it is easier to push the negative ion out of the surface than the positive one. We conclude that the negative ion cannot have the same or a more massive structure than the positive, and therefore the structures I and IV must be discarded. We stress the practical importance of discarding IV because that means that in liquid helium the concentration of impurities is really negligible, or at least that they do not contribute at all in these experiments with ions.

Next we consider the absence of extraction currents for negative ions in the low temperature runs. This indicates that for negative ions tunnelling effects are negligible, an important result which rules out its possible structure III.

Finally the solid helium experiment shows that for the positive ion the structure II involving jumps eventually by tunnelling must also be discarded.

5. – Conclusion.

The only structures left are then I for the positive ion and II for the negative ion, or briefly the positive ion is a cluster and the negative is an empty bubble. We believe this to be in agreement with all other experimental information concerning the mobility, since the mobility of the negative ions is lower than the positive ones [1]. This is due to the larger extension of the shell surrounding the negative cloud in comparison with the solid droplet surrounding the positive charge, as indicated in 1.

It is hoped that future theoretical and experimental work may lay better foundations on the structures, which have been adopted here on the basis of the experimental evidence so far at hand.

RIASSUNTO

Sono stati eseguiti degli esperimenti per scegliere un modello adatto fra alcune strutture possibili per gli ioni di elio in elio liquido. L'apparecchio è essenzialmente un diodo, dove gli ioni sono prodotti da raggi α e le correnti possono essere misurate in fase liquida, solida e in interfase liquido-vapore. Tramite il differente comportamento degli ioni positivi e negativi negli esperimenti in interfase, si è portati a ritenere lo ione positivo come un raggruppamento di atomi polarizzati intorno ad una carica, e lo ione negativo come una cavità dove l'elettrone si auto-intrappola.

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Über die Mannigfaltigkeit der interpolierenden Felder zu einer kausalen S-Matrix.

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(ricevuto il 21 Dicembre 1959)

Summary. — Local commutativity of different fields is shown to be a transitive property of fields. A criterion is derived which permits to decide whether two local fields are interpolating fields to the same S-Matrix or not. It is shown that there exist more than one interpolating field to a given causal S-Matrix.

1. – Einleitung.

Man nennt eine S-Matrix kausal (1), wenn zu den freien Feldern $A_{in}(x)$ und $A_{out}(x)$ ein lokales, interpolierendes Feld $A(x)$ existiert, welches asymptotisch in $A_{in}(x)$ bzw. in $A_{out}(x)$ übergeht (2). Da die Frage nach der Existenz einer nicht trivialen S-Matrix bis jetzt noch ungeklärt ist, wird es von Interesse sein, sich über die Mannigfaltigkeit der interpolierenden Felder einer S-Matrix Klarheit zu verschaffen. Dabei wird die Existenz einer kausalen S-Matrix und damit auch die Existenz mindenstens eines lokalen, interpolierenden Feldes vorausgesetzt.

Es wird zunächst ein Kriterium dafür angegeben, wann zwei lokale Felder zur gleichen S-Matrix gehören. Dann wird eine Klasseneinteilung der möglichen interpolierenden Felder vorgenommen, wozu man die Transitivität der Lokalität beweisen muß. Es wird gezeigt, daß jede Klasse aus untereinander lokal vertauschbaren Feldern besteht, und durch explizite Konstruktion beweist man

(1) Siehe hierzu H. Lehmann, K. Symanzik und W. Zimmermann: *Nuovo Cimento*, **6**, 319 (1957).

(2) Zur Formulierung der Asymptotenbedingung vergl. H. Lehmann, K. Symanzik und W. Zimmermann: *Nuovo Cimento*, **1**, 425 (1955), zum Beweis derselben l. c. (8).

weiter, daß jede Klasse mehrere Felder als Elemente enthält. Ungeklärt bleibt dagegen, ob wirklich mehrere Klassen zu einer kausalen S-Matrix existieren, oder ob schon eine Klasse die Gesamtheit aller lokalen, interpolierenden Felder erschöpft.

Zur Vereinfachung des Problems betrachten wir den Fall neutraler, selbstwechselwirkender Felder ohne gebundene Zustände, die durch hermitische Operatoren $A(x)$, $B(x)$, ... beschrieben werden. Weiter wird angenommen, daß die Felder bei inhomogenen Lorentztransformationen invariant sind:

$$U(L)A(x)U^{-1}(L) = A(Lx) .$$

Die unitären Operatoren $U(L)$ sollen für alle Felder $A(x)$, $B(x)$, ... die gleichen sein. Das Energie- und Massenspektrum soll nicht negativ sein, letzteres soll abgesehen vom Vakuumzustand den diskreten Eigenwert m und kontinuierliche Eigenwerte von $2m$ bis ∞ besitzen. Weiter sollen die Felder lokal sein

$$\text{(1)} \qquad \begin{cases} [A(x), A(y)] = 0 \\ [B(x), B(y)] = 0 \end{cases} \qquad \text{für } x - y \text{ raumartig,}$$

und asymptotische Felder besitzen [3]

$$\text{(2)} \qquad \begin{cases} A_{\text{in}}(x) = A(x) + \int \Delta_r(x - x') j_A(x')\, \mathrm{d}x' , \\ B_{\text{in}}(x) = B(x) + \int \Delta_r(x - x') j_B(x')\, \mathrm{d}x' , \end{cases}$$

mit

$$j_A(x) = (\Box_x - m^2) A(x) .$$

Die auslaufenden Felder werden entsprechend mit der avancierten Funktion definiert. Schließlich wird angenommen, daß die einlaufenden Felder vollständig sind.

2. – Transitivität der Lokalität.

Wir beginnen zunächst mit eigenen Vorbemerkungen zum TCP-Operator Θ. Sei Θ' ein antiunitärer Operator mit der Eigenschaft

$$\text{(3)} \qquad \Theta' A(x) \Theta'^{-1} = A(-x) ,$$

[3] Die Gleichungen (2) sind in Sinne schwacher Konvergenz zu verstehen.

wobei $A(x)$ ein vollständiges hermitisches Feld ist, so folgt durch Iteration von (3):

$$\Theta'\Theta' = \lambda \qquad \text{mit } |\lambda| = 1.$$

Aus

$$\Theta'\Theta'\Theta' = \begin{cases} (\Theta'\Theta')\Theta' = \lambda\Theta' \\ \Theta'(\Theta'\Theta') = \lambda^*\Theta' \end{cases}$$

folgt $\lambda = \pm 1$. Fordert man, daß Θ' den Vakuumzustand in ein Vielfaches des Vakuumzustandes überführt, und daß der Vakuumzustand nicht entartet ist, so erkennt man, daß $\lambda = 1$ als einzige Möglichkeit nachbleibt, also:

$$\Theta'\Theta' = 1 . \tag{4}$$

Sind nun Θ'_1 und Θ'_2 zwei antiunitäre Operatoren mit der Eigenschaft (3), so folgt in analoger Schlußweise:

$$\Theta'_1\Theta'_2 = \lambda , \qquad \Theta'_2\Theta'_1 = \lambda^* \qquad \text{mit } |\lambda| = 1.$$

Durch die Forderung (3) ist der Operator Θ' also bis auf einen Phasenfaktor eindeutig durch das Feld $A(x)$ bestimmt, den TCP-Operator Θ erhält man durch Wahl der Phase, und zwar derart, daß Θ den Vakuumzustand reproduziert:

$$\Theta\Omega = \Omega . \tag{5}$$

Θ ist also durch $A(x)$ eindeutig bestimmt .

Transformiert man Gleichung (2) mit Θ, so erhält man wegen $\Delta_r(x - x') = \Delta_a(x' - x)$

$$\Theta A_{\text{in}}(x)\Theta = A_{\text{out}}(-x) . \tag{6}$$

Analog den obigen Überlegungen folgt aus der Vollständigkeit der einlaufenden bzw. auslaufenden Felder, daß der TCP-Operator auch eindeutig durch die asymptotischen Felder $A_{\text{in}}(x)$ und $A_{\text{out}}(x)$ festgelegt ist.

Notwendig und hinreichend für die Existenz eines TCP-Operators Θ ist nach R. Jost [4] die schwache Lokalität. Unter einem schwach lokalen Feld verstehen wir ein Feld, dessen zugehörige Wightman-Funktionen [5] in re:llen

[4] R. Jost: *Helv. Phys. Acta*, **30**, 409 (1957).

[5] Über die analytischen Eigenschaften dieser Funktionen siehe A. S. Wightman: *Phys. Rev.*, **101**, 860 (1956).

Regularitätspunkten (6) die Symmetrie

$$(7) \qquad (\Omega, A(x_1)A(x_2)\dots A(x_n)\Omega) = (\Omega, A(x_n)\dots A(x_2)A(x_1)\Omega)$$

besitzen. Sind mehrere Felder vorhanden, so ist notwendig und hinreichend für die Existenz eines gemeinsamen TCP-Operators auch wieder die schwache Lokalität aller Felder untereinander, d.h.

$$(7') \qquad (\Omega, A(x_1)B(x_2)\dots C(x_n)\Omega) = (\Omega, C(x_n)\dots B(x_2)A(x_1)\Omega)$$

gilt für die reellen Regularitätspunkte der Wightman-Funktionen.

Jetzt untersuchen wir die Transitivitätseigenschaften der Lokalität, und zwar zeigen wir:

I. Ist $A(x)$ schwach lokal und vollständig, und gilt

$$(\Omega, A(x_1)\dots A(x_{i-1})B(x_i)A(x_{i+1})\dots A(x_n)\Omega) = \\ = (\Omega, A(x_n)\dots A(x_{i+1})B(x_i)A(x_{i-1})\dots A(x_1)\Omega)$$

für alle i und n, $(i \leqslant n)$ in den reellen Regularitätspunkten, so ist

a) $B(x)$ ein schwach lokales Feld, und

b) $A(x)$ und $B(x)$ sind untereinander schwach lokal.

II. Sind $A(x)$, $B(x)$, $C(x)$ schwach lokal, ist $A(x)$ vollständig und $A(x)$ mit $B(x)$, sowie $A(x)$ mit $C(x)$ schwach lokal, so ist auch $B(x)$ mit $C(x)$ schwach lokal.

III. Ist $A(x)$ lokal (im Sinne von Gl. (1)) und vollständig, und gilt

$$[A(x), B(y)] = 0 \qquad \text{für } x - y \text{ raumartig},$$

so ist $B(x)$ ein lokales Feld.

IV. Ist $A(x)$ lokal und vollständig, sind $B(x)$, $C(x)$ lokal und gilt

$$[A(x), B(y)] = [A(x), C(y)] = 0 \quad \text{für } x - y \text{ raumartig},$$

so gilt auch

$$[B(x), C(x)] = 0 \qquad \text{für } x - y \text{ raumartig}.$$

(6) Reelle Regularitätspunkte sind immer total raumartige Punkte. Die Umkehrung gilt für lokale Felder. Siehe D. RUELLE: *Helv. Phys. Acta*, **32**, 135 (1959).

Zum Beweise von I. gehen wir aus von der Gleichung

$$(\Theta\Psi,\ \Theta B(x)\Theta\Theta\Phi) = (\Psi,\ B(x)\Phi)^* = (\Phi,\ B(x)\Psi) \tag{8}$$

wobei Θ der zum Felde $A(x)$ gehörige TCP-Operator ist

Mit

$$\Psi = \sum_k \int \mathrm{d}x_1 \dots \mathrm{d}x_k f_k(x_1 \dots x_k)\, A(x_1) \dots A(x_k)\Omega)$$

$$\Phi = \sum_l \int \mathrm{d}x_1 \dots \mathrm{d}x_l g_l(x_1 \dots x_l)\, A(x_1) \dots A(x_l)\Omega)$$

$$\Theta\Psi = \sum_k \int \mathrm{d}x_1 \dots \mathrm{d}x_k f_k^*(x_1 \dots x_k)\, A(-x_1) \dots A(-x_k)\Omega)$$

$$\Theta\Phi = \sum_l \int \mathrm{d}x_1 \dots \mathrm{d}x_l g_l^*(x_1 \dots x_l)\, A(-x_1) \dots A(-x_l)\Omega)$$

folgt andererseits aus dem TCP-Theorem und den Voraussetzungen von I.:

$$\begin{aligned}(\Phi,\ B(z)\Psi) &= \sum_{kl} \int f_k(x_1 \dots x_k) g_l^*(y_1 \dots y_l) \cdot \\ &\quad \cdot (\Omega,\ A(y_l) \dots A(y_1) B(z) A(x_1) \dots A(x_k)\Omega)\, \mathrm{d}x_1 \dots \mathrm{d}x_k\, \mathrm{d}y_1 \dots \mathrm{d}y_l\,, \\ &= \sum_{kl} \int f_k(x_1 \dots x_k) g_l^*(y_1 \dots y_l) \cdot \\ &\quad \cdot (\Omega,\ A(-x_k) \dots A(-x_1) B(-z) A(-y_1) \dots A(-y_l)\Omega)\, \mathrm{d}x_1 \dots \mathrm{d}x_k\, \mathrm{d}y_1 \dots \mathrm{d}y_l\,, \\ (\Phi,\ B(z)\Psi) &= (\Theta\Psi,\ B(-z)\,\Theta\Phi)\,.\end{aligned} \tag{9}$$

Ein Vergleich von (8) mit (9) liefert

$$\Theta B(x)\Theta = B(-x)\,, \tag{10}$$

d.h., zum Felde $B(x)$ existiert ein TCP-Operator, und dieser ist mit dem TCP-Operator des Feldes $A(x)$ identisch. Auf Grund des TCP-Theorem ist diese Aussage zur Behauptung I. äquivalent.

II. ist sehr einfach, da auf Grund des TCP-Theorems aus den Voraussetzungen folgt, daß sowohl $A(x)$ und $B(x)$ als auch $A(x)$ und $C(x)$ einen gemeinsamen TCP-Operator besitzen. Da nun $A(x)$ vollständig ist, ist dieser Operator eindeutig, also haben auch $B(x)$ und $C(x)$ einen gemeinsamen TCP-Operator. Eine erneute Anwendung des TCP-Theorems liefert II.

Um III. zu beweisen, stellen wir zunächst fest, daß die Felder $A(x)$ und $B(x)$ die Voraussetzungen von I. erfüllen und somit schwach lokal sind. Da

nun $A(x)$ lokal ist und $A(x)$ mit $B(y)$ für raumartige Abstände vertauscht, gilt für die reellen Regularitätspunkte der Wightman-Funktionen:

$$(11)\qquad (\Omega, A(x_1)\dots A(x_i)B(y_1)B(y_2)A(x_{i+1})\dots A(x_n)\Omega) = \\ = (\Omega, A(x_n)\dots A(x_{i+1})B(y_2)B(y_1)A(x_i)\dots A(x_1)\Omega) = \\ = (\Omega, A(x_1)\dots A(x_i)B(y_2)B(y_1)A(x_{i+1})\dots A(x_n)\Omega)\,.$$

Hieraus läßt sich die Lokalität von $B(x)$ erschließen. Wir führen den Beweis jedoch gemeinsam mit dem analogen Beweis von IV.

Zunächst folgt aus den Voraussetzungen von IV., daß nach II. $A(x)$, $B(x)$ und $C(x)$ untereinander schwach lokal sind. Da nun $A(x)$ lokal ist und mit $B(y)$ und $C(y)$ für raumartige Abstände vertauscht, gilt für die reellen Regularitätspunkte der Wightman-Funktionen:

$$(12)\qquad (\Omega, A(x_1)\dots A(x_i)B(y_1)C(y_2)A(x_{i+1})\dots A(x_n)\Omega) = \\ = (\Omega, (Ax_n)\dots A(x_{i+1})C(y_2)B(y_1)A(x_i)\dots A(x_1)\Omega) = \\ = (\Omega, A(x_1)\dots A(x_i)C(y_2)B(y_1)A(x_{i+1})\dots A(x_n)\Omega)\,.$$

Die Gleichung (11) erhalten wir aus (12), indem wir $C(y_2) = B(y_2)$ setzen. Ebenso erhält man den zweiten Teil des Beweises von III. aus dem Folgenden durch die gleiche Ersetzung.

Bekanntlich sind die Wightman-Funktionen [6]

$$(13)\qquad W^{(1)}(x_1-x_2, \dots, x_i-y_1, y_1-y_2, y_2-x_{i+1}, \dots, x_{n-1}-x_n) = \\ = (\Omega, A(x_1)\dots A(x_i)B(y_1)C(y_2)A(x_{i+1})\dots A(x_n)\Omega) = \\ = W^{(1)}(\xi_1, \dots, \xi_i, \eta, \xi_{i+1}, \dots, \xi_n) \qquad \eta = y_1-y_2.$$

Randwerte analytischer Funktionen

$$W^{(1)}(\xi_1, \dots, \xi_i, \eta, \xi_{i+1}, \dots, \xi_n) = \lim_{\substack{\xi_k'\eta' \to 0 \\ \xi_k'\eta' \in L^+}} W^{(1)}(\xi_i+i\xi_1', \dots, \eta+i\eta', \dots, \xi_n+i\xi_n')\,.$$

In analoger Weise ist auch

$$(13')\qquad W^{(2)}(\xi_1, \dots, \xi_{i-1}, \xi_i+\eta, -\eta, \xi_{i+1}+\eta, \dots, \xi_n) = \\ = (\Omega, A(x_1)\dots C(y_2)B(y_1)\dots A(x_n)\Omega)\,.$$

Randwert einer analytischen Funktion. Daraus folgt, daß

$$F(\eta; \xi_1\dots\xi_n) = W^{(1)}(\xi_1\dots\xi_i, \eta, \xi_{i+1}\dots\xi_n) - W^{(2)}(\xi_1, \dots, \xi_i+\eta, -\eta, \xi_{i+1}+\eta, \dots, \xi_n)$$

bei festem η auch Randwert einer analytischen Funktion

$$F(\eta;\ \xi_1+i\xi_1', \ldots,\ \xi_n-i\xi_n'), \qquad \xi_k'\in L^+$$

ist. Wählen wir nun η raumartig, so gilt für die total raumartigen Punkte ([6]) aller x und y:

$$W^{(1)}(\xi_1 \ldots \xi_i, \eta,\ \xi_{i+1}\ldots\xi_n) = W^{(2)}(\xi_1\ldots\xi_i+\eta, -\eta, \xi_{i+1}+\eta, \ldots, \xi_n),$$

also:

$$F(\eta;\ \xi_1\ldots\xi_n) = 0.$$

Ebenso verschwindet auch der Randwert von $F^*(\eta;\ \xi_1-i\xi_1', \ldots, \xi_n-i\xi_n')$ in in den gleichen Punkten. Nach dem « Edge of the Wedge »-Theorem ([7]) folgt, daß $F(\eta;\ \xi_1\ldots\xi_n)$ in diesen Punkten regulär ist, und also dort verschwindet. Daraus folgt, daß die ganze analytische Funktion verschwindet, und somit auch ihre Randwerte, d.h.:

$$W^{(1)}(\xi_1\ldots\xi_i,\ \eta,\ \xi_{i+1}\ldots\xi_n) = W^{(2)}(\xi_1\ldots,\ \xi_i+\eta, -\eta,\ \xi_{i+1}+\eta, \ldots\ \xi_n) \quad \text{für } \eta \text{ raumartig.}$$

Wegen der Vollständigkeit des Feldes $A(x)$ folgt daraus nach (13) und (13′)

$$[B(y_1),\ C(y_2)] = 0 \qquad \text{für } y_1-y_2 \text{ raumartig,}$$

und mit der Ersetzung $C(y_2) = B(y_2)$

$$[B(y_1),\ B(y_2)] = 0 \qquad \text{für } y_1-y_2 \text{ raumartig.}$$

Damit ist IV. bzw. III. bewiesen.

3. – Über die Mannigfaltigkeit der interpolierenden Felder einer S-Matrix.

Wir wollen jetzt die interpolierender Felder zu einer vorgegebenen kausalen S-Matrix untersuchen. Da die S-Matrix kausal ist, existiert also mindestens ein lokales interpolierendes Feld.

Zunächst fragen wir nach einem Kriterium, welches zu entscheiden gestattet, ob ein weiteres vorgelegtes Feld die gleiche S-Matrix interpoliert oder nicht. Dazu zeigen wir:

([7]) F. J. Dyson: *Phys. Rev.*, **110**, 579 (1958), siehe dort auch weitere Hinweise.

V. Sind $A(x)$ und $B(x)$ zwei lokale Felder, und ist

$$A_{\text{in}}(x) = B_{\text{in}}(x)\,,$$

so ist die schwache Lokalität von $A(x)$ mit $B(x)$ notwendig und hinreichend dafür, daß beide Felder interpolierende Felder zur gleichen S-Matrix sind.

Zum Beweise von V. nehmen wir zunächst an, daß $A(x)$ und $B(x)$ schwach lokal sind; dann besitzen nach dem TCP-Theorem beide Felder den gleichen TCP-Operator Θ. Da nun $\Theta A_{\text{in}}(x)\,\Theta = A_{\text{out}}(-x)$ und $\Theta B_{\text{in}}(x)\,\Theta = B_{\text{out}}(-x)$ ist, folgt aus $A_{\text{in}}(x) = B_{\text{in}}(x)$ auch

$$A_{\text{out}}(x) = B_{\text{out}}(x)\,,$$

und somit die Gleichheit der S-Matrizen.

Sind nun andererseits $A(x)$ und $B(x)$ interpolierende Felder zur gleichen S-Matrix, so ist mit $A_{\text{in}}(x) = B_{\text{in}}(x)$ auch $A_{\text{out}}(x) = B_{\text{out}}(x)$. Da $A(x)$ und $B(x)$ lokale Felder sind, existieren die TCP-Operatoren Θ_A und Θ_B. Diese sind aber eindeutig durch die ein- und auslaufenden Felder bestimmt. Da diese aber übereinstimmen, sind auch die beiden TCP-Operatoren Θ_A und Θ_B identisch. Nach dem TCP-Theorem folgt daraus, dad $A(x)$ und $B(x)$ schwach vertauschbar sind.

Auf Grund der in IV. bewiesenen Transitivität der Lokalität bilden jeweils die miteinander stark vertauschbaren Felder in der Menge der lokalen, interpolierenden Felder einer S-Matrix eine Klasse. Wie sich zeigen läßt, sind alle miteinander stark vertauschbaren Felder Mitglieder einer Klasse, ohne daß man dabei die Identität der asymptotischen Felder voraussetzen muß. Dabei ist allerdings eine Einschränkung zu machen, ist $A(x)$ mit $B(x)$ lokal vertauschbar, so ist entweder $B(x)$ oder $-B(x)$ mit $A(x)$ in einer Klasse. Zum Beweise zeigen wir:

VI. Sind $A(x)$, $B(x)$ lokal, $A(x)$ vollständig, gilt

$$[A(x), B(y)] = 0 \qquad \text{für } x - y \text{ raumartig},$$

und haben $A(x)$ und $B(x)$ asymptotische Felder zur gleichen Masse, so ist $B_{\text{in}}(x) = \pm A_{\text{in}}(x)$.

Den Beweis von VI. führen wir nach der Methode von W. ZIMMERMANN (8).

(8) W. ZIMMERMANN: *Nuovo Cimento*, **10**, 597 (1958).

Dazu gehen wir aus von der Identität [9,10]:

$$\int \mathrm{d}x \int \mathrm{d}y\, f_\alpha^*(x)\, f_\beta(y)\, K_x K_y\, TA(x)B(y) = \int \mathrm{d}y \int \mathrm{d}x\, f_\alpha^*(x)\, f_\beta(y)\, K_x K_y\, TA(x)B(y)\,. \tag{14}$$

Dabei ist $f_\alpha(x)$ ein vollständiges Orthogonalsystem mit positiven Frequenzen der Lösungen $(\Box_x - m)f(x) = 0$. Führen wir eine Integration auf der linken Seite von (14) aus, so erhalten wir mit Hilfe des Green'schen Satzes:

$$-i\int \mathrm{d}y\, f_\beta(y)\, K_y\, TA(x)B(y) = -i\int \mathrm{d}y_0 \frac{\partial}{\partial y_0} \int \mathrm{d}^3y\, TA(x)B(y) \overleftrightarrow{\frac{\partial}{\partial y_0}} f_\beta(y) =$$

$$= A(x)B_{\mathrm{in}\,\beta}^- - B_{\mathrm{out}\,\beta}^- A(x)$$

mit

$$f(x) \overleftrightarrow{\frac{\partial}{\partial y_0}} g(x) = f(x)\frac{\partial g(x)}{\partial x_0} - g(x)\frac{\partial f(x)}{\partial x_0}\,.$$

Führen wir die Integration über x in gleicher Weise aus, so erhalten wir:

$$\int \mathrm{d}x \int \mathrm{d}y\, f_\alpha^*(x)\, f_\beta(y)\, K_x K_y\, TA(x)B(y) = B_{\mathrm{out}\,\beta}^- A_{\mathrm{out}\,\alpha}^+ - B_{\mathrm{out}\,\beta}^- A_{\mathrm{in}\,\alpha}^+ - A_{\mathrm{out}\,\alpha}^+ B_{\mathrm{in}\,\beta}^- + A_{\mathrm{in}\,\alpha}^+ B_{\mathrm{in}\,\beta}^-\,.$$

Für das Integral auf der rechten Seite von (14) erhalten wir entsprechend:

$$\int \mathrm{d}y \int \mathrm{d}x\, f_\alpha^*(x)\, f_\beta(y)\, K_x K_y\, TA(x)B(y) = A_{\mathrm{out}\,\alpha}^+ B_{\mathrm{out}\,\beta}^- - A_{\mathrm{out}\,\alpha}^+ B_{\mathrm{in}\,\beta}^- - B_{\mathrm{out}\,\beta}^- A_{\mathrm{in}\,\alpha}^+ + B_{\mathrm{in}\,\beta}^- A_{\mathrm{in}\,\alpha}^+\,.$$

Ein Vergleich der beiden Integrale liefert:

$$[B_{\mathrm{out}\,\beta}^-,\, A_{\mathrm{out}\,\alpha}^+] = [B_{\mathrm{in}\,\beta}^-,\, A_{\mathrm{in}\,\alpha}^+]\,.$$

Mit den entsprechenden Gleichungen zwischen $B_{\mathrm{out}}^+_{\mathrm{in}}$ und $A_{\mathrm{out}}^+_{\mathrm{in}}$ usw. folgt wegen der Vollständigkeit des Systems $f_\alpha(x)$

$$[A_{\mathrm{in}}(x),\, B_{\mathrm{in}}(y)] = [A_{\mathrm{out}}(x),\, B_{\mathrm{out}}(y)]\,. \tag{15}$$

[9] Zum Beweis der Identitäten (14) und (16) siehe W. Zimmermann. *Order of integrations in reduction formulae*, unveröffentlichtes Manuskript.

[10] Die Gleichungen (14) und (16) sind im Sinne schwacher Konvergenz zu lesen.

Zum weiteren Beweis gehen wir aus von der Identität

$$(16)\qquad \int dx \int dy\, f_\alpha^*(x)\, f_\beta(y)\, K_x K_y\, TA(x)\, B(y)\, A(z) = \\ = \int dy \int dx\, f_\alpha^*(x)\, f_\beta(y)\, K_x K_y\, TA(x)\, B(y)\, A(z)\,,$$

die mit Hilfe des Green'schen Satzes

$$A(z)\,[A^+_{\mathrm{in}\,\alpha},\, B^-_{\mathrm{in}\,\beta}] = [A^+_{\mathrm{out}\,\alpha},\, B^-_{\mathrm{out}\,\beta}]A(z)\,,$$

ergibt. Zusammen mit (15) erhält man:

$$\big[[A^+_{\mathrm{in}\,\alpha},\, B^-_{\mathrm{in}\,\beta}],\, A(z)\big] = 0\,.$$

Mit den entsprechenden Gleichungen für $A^-_{\mathrm{in}\,\alpha}$ und $B^+_{\mathrm{in}\,\beta}$ findet man:

$$(17)\qquad \big[[A_{\mathrm{in}}(x),\, B_{\mathrm{in}}(y)],\, A(z)\big] = 0\,.$$

Wegen der Vollständigkeit des Feldes $A(z)$ erhält man, daß der Kommutator der beiden einlaufenden Felder eine c-Zahl ist:

$$[A_{\mathrm{in}}(x),\, B_{\mathrm{in}}(y)] = (\Omega,\, [A_{\mathrm{in}}(x),\, B_{\mathrm{in}}(y)]\Omega) = F(y-x)\,.$$

Da nun $A(x)$ und $B(x)$ einen gemeinsamen TCP-Operator haben, folgt zusammen mit (16):

$$-F(x-y) = (\Omega,\, [(B_{\mathrm{in}}(y),\, A_{\mathrm{in}}(x)]\Omega) = (\Omega,\, [A_{\mathrm{in}}(x),\, B_{\mathrm{in}}(y)]\,\Omega)^* = \\ = (\Theta\Omega,\, \Theta\,[A_{\mathrm{in}}(x),\, B_{\mathrm{in}}(y)]\Omega) = (\Omega,\, [A_{\mathrm{out}}(-x),\, B_{\mathrm{out}}(-y)]\Omega) = \\ = (\Omega,\, [A_{\mathrm{in}}(-x),\, B_{\mathrm{in}}(-y)]\Omega) = F(-x+y)\,;$$

d.h., $F(x-y)$ muß für raumartige $(x-y)$ verschwinden, und somit ist:

$$([A_{\mathrm{in}}(x),\, B_{\mathrm{in}}(y)]\Omega) = \lambda \frac{1}{i}\, \Delta(x-y)\,.$$

Aus der Hermitizität der Felder folgt $\lambda^* = \lambda$ also $\lambda = \pm 1$.
Aus

$$[A_{\mathrm{in}}(x),\, B_{\mathrm{in}}(y)] = \pm \frac{1}{i}\, \Delta(x-y)\,.$$

folgt nun aber

$$B_{\text{in}}(x) = \pm A_{\text{in}}(x) .$$

Zum Abschluß wollen wir an einem Beispiel zeigen, daß jede Klasse mehrere Felder enthält. Ist die S-Matrix gleich der Einheitsmatrix, so ist mit dem freien Feld auch das Feld [11]

$$A(x) = A_{\text{in}}(x) + : A_{\text{in}}^2(x) :$$

ein interpolierendes Feld. Ist $S \neq 1$, so ist $K_x A(x) \neq 0$. Demnach ist mit $A(x)$ auch

$$B(x) = A(x) + K_x A(x)$$

ein interpolierendes Feld zur gleichen S-Matrix. Aus der Existenz des asymptotischen Feldes und der Forderung, daß $A_{\text{in}}(x)$ vollständig ist, folgt, daß auch $B(x)$ vollständig ist. Allgemein gehören

$$B(x) = A(x) + P(\Box_x) K_x A(x)$$

zur gleichen Klasse wie $A(x)$. P ist dabei ein reelles Polynom.

Die Anzahl der Elemente einer Klasse ist also mindestens von der Mächtigkeit der reellen Polynome in einer Variablen.

* * *

Die Arbeit geht zurück auf Untersuchungen, die dem Verfasser durch ein Stipendium der Deutschen Forschungsgemeinschaft ermöglicht wurden. Der Verfasser möchte hier der Deutschen Forschungsgemeinschaft seinen Dank aussprechen.

[11] Vergl. M. A. WIGHTMAN: *Problèmes Mathématiques de la Théorie Quantique des Champs* (Paris, 1958).

RIASSUNTO (*)

Si dimostra che la proprietà commutativa locale di campi diversi è una proprietà transitiva dei campi. Si deduce un criterio che permette di decidere se due campi locali sono campi interpolanti nella stessa matrice S o no. Si dimostra che esiste più di un campo interpolante per ogni data matrice S causale.

(*) *Traduzione a cura della Redazione.*

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Diffusion of ^{37}A, Kr, HT in Liquid Argon Between (84÷90) °K.

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(ricevuto il 30 Dicembre 1959)

Summary. — The diffusion of tracers in liquid argon have been measured in a wide range of temperatures and under a pressure of 2 atm. The results can be expressed by the following equations:

$$D(\text{HT-A}) = 110 \quad \exp[-312/T]\cdot 10^{-5} \text{ cm}^2/\text{s} ,$$

$$D(^{37}\text{A-A}) = 61 \quad \exp[-312/T]\cdot 10^{-5} \text{ cm}^2/\text{s} ,$$

$$D(\text{Kr-A}) = 52.5 \exp[-312/T]\cdot 10^{-5} \text{ cm}^2/\text{s} .$$

The results of the Brownian motion approach to the transport phenomena are compared with the experimental data on self-diffusion. Binary diffusions are instead discussed on the basis of phenomenological treatments.

1. – Introduction.

Transport processes occur in fluids in non uniform states. The fluxes of mass, momentum and energy in normal liquids near equilibrium are proportional to the gradients of concentration, velocity and temperature, the respective proportionality constants being the coefficients of diffusion, viscosity and thermal conductivity.

The connection between the phenomenological transport coefficients and the microscopic structure of the liquid is the first object of the investigations in this field. An analysis of the molecular scale process of diffusion seems to be a fruitful approach to the solution of this problem. In fact the diffusion coefficient D is related to the molecular parameters of the diffusing molecules [1]

[1] G. Cini-Castagnoli, G. Pizzella and F. P. Ricci: *Nuovo Cimento*, **10**, 300 (1958).

and is connected, through the Einstein equation, to the frictional coefficient ζ whose determination is the principal problem of the Brownian motion theory [2] of liquid transport processes. Moreover a precise determination of D allows to test and to understand the simpler relationships, which involve directly measurable quantities and the transport coefficients, as the Stokes-Einstein relation.

For these reasons we have investigated the self diffusion and the diffusion of tracers in liquid argon; in fact Argon, Kripton and Tritium are very simple molecules and all the considerations stated above can be applied to them.

2. – Apparatus and experimental results.

2·1. *Apparatus.* – A schematic representation of the diffusion apparatus used for this investigation is shown in Fig. 1. It consists essentially of a capil-

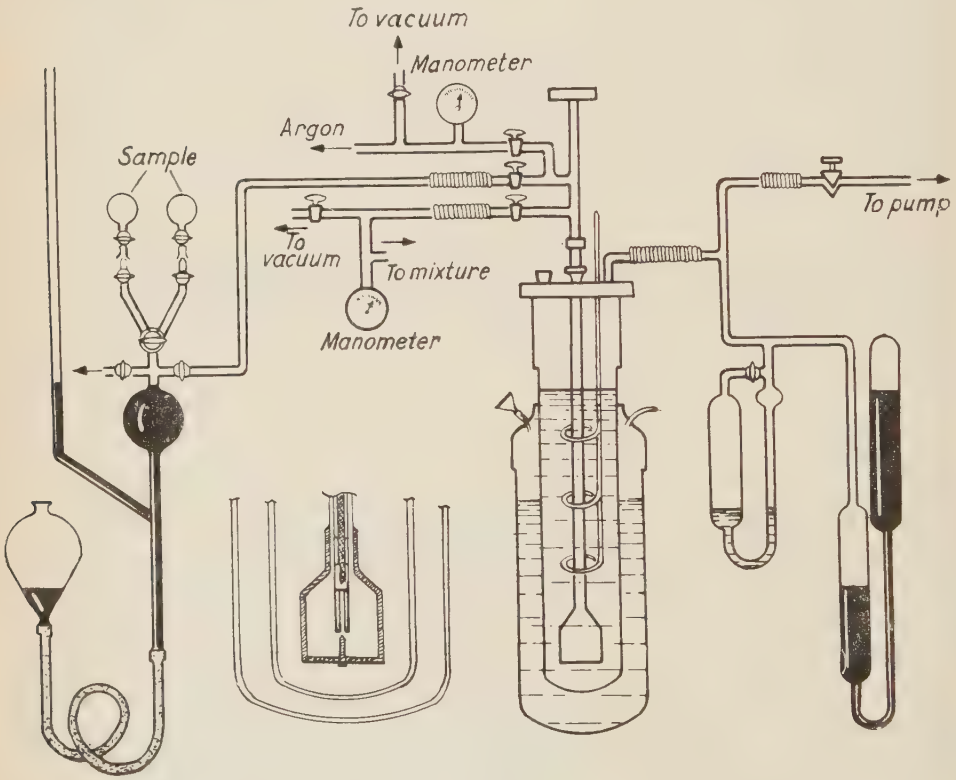

Fig. 1. – Experimental apparatus.

[2] J. G. KIRKWOOD: *Journ. Chem. Phys.*, **14**, 180 (1946).

lary in contact with an infinite bath, suitable to the work at low temperatures and to the condensation of gases under pressure. Namely, as the temperature range in which the argon is liquid below atmospheric pressure is of the order of only three degrees, we thought that it was necessary to extend such an interval by increasing the pressure, in order to determine with good accuracy the temperature dependence of the diffusion coefficient. Therefore we have modified an apparatus, previously described ([1]), for work under a pressure of (2÷3) atmospheres. The main characteristic of this new apparatus consists in the separatiion of the capillary filling system (under pressure of 2 atm) from the sample removing system (Töpler pump working at a lower pressure). The range of temperatures obtained in this way makes it possible to investigate the diffusion coefficient of ^{37}A, HT, Kr, in liquid argon between 84 °K and and 90 °K. The low temperature was provided by means of a bath of liquid oxygen on which the pressure was controlled within 0.5 mm Hg, so that the temperature was kept constant within less then 0.1 °K during the experiments.

The capillaries dimensions (diameter d and length L) were chosen so that, with good approximation, a one-dimensional diffusion proceeds in the overall length of the capillary (*i.e.* $d/L < 0.025$). In this way the effects of the deformations at the lower end of the capillary during the sealing operations are negligible.

The details of the experimental procedure and of sample concentration measurements (performed with G.M. counters) have been fully described in previous papers ([1,3,4]). The tracer's concentrations in pure argon of the starting mixtures were of the order of 10^{-7}%.

2·2. *Experimental results.* – The diffusion coefficient D can be evaluated ([4]) from the knowledge of the average concentration in the capillary after the diffusion, from the concentration of the mixture in the bath and from the

TABLE I. – *Experimental results for* Kr-A *diffusion.*

run no.	diameter d (cm)	lenght L (cm)	time t (s)	temperature (°K)	D ($\cdot 10^5$ cm^2/s)
1	0.04	2.30±0.02	15300	90.20±0.05	1.72±0.04
2	0.04	2.30±0.02	6300	90.20±0.05	1.62±0.05
3	0.04	2.30±0.02	15000	87.94±0.05	1.54±0.08
4	0.04	2.30±0.02	9000	86.38±0.05	1.38±0.06
5	0.04	2.30±0.02	13500	86.10±0.05	1.40±0.10
6	0.04	2.30±0.02	16680	85.44±0.05	1.35±0.10

([3]) G. CINI-CASTAGNOLI, A. GIARDINI and F. P. RICCI: *Nuovo Cimento*, **13**, 916 (1959).
([4]) G. CINI-CASTAGNOLI and F. P. RICCI: *Journ. Chem. Phys.*, **32**, 19 (1960).

TABLE II. – *Experimental results for* ^{37}A-A *diffusion.*

run no.	diameter d (cm)	lenght L (cm)	time t (s)	temperature (°K)	D ($\cdot 10^5$ cm^2/s)
1	0.05	2.34 ± 0.03	6090	90.10 ± 0.05	2.10 ± 0.10
2	0.05	2.34 ± 0.03	4680	90.10 ± 0.05	1.89 ± 0.08
3	0.05	2.49 ± 0.03	7860	86.96 ± 0.05	1.71 ± 0.10
4	0.05	2.49 ± 0.03	3800	87.12 ± 0.05	1.71 ± 0.10
5	0.05	2.34 ± 0.03	12300	87.98 ± 0.05	1.73 ± 0.10
see ref. (4)	—	—	—	84.56 ± 0.04	1.53 ± 0.03

TABLE III. – *Experimental results for* HT-A *diffusion.*

run no.	diameter d (cm)	lenght L (cm)	time t (s)	temperature (°K)	D ($\cdot 10^5$ cm^2/s)
1	0.04	2.30 ± 0.02	6360	85.17 ± 0.05	2.69 ± 0.10
2 (*)	0.04	2.30 ± 0.02	11400	85.12 ± 0.05	2.91 ± 0.10
3	0.04	2.30 ± 0.02	9000	90.11 ± 0.05	3.36 ± 0.10
4	0.04	2.30 ± 0.02	3600	90.11 ± 0.05	3.38 ± 0.10
5	0.04	2.30 ± 0.02	9600	87.24 ± 0.05	3.01 ± 0.14
6	0.04	2.30 ± 0.02	3600	87.26 ± 0.05	3.02 ± 0.16
7	0.04	2.30 ± 0.02	14400	88.01 ± 0.05	3.23 ± 0.04
8	0.04	2.30 ± 0.02	4800	88.01 ± 0.05	3.21 ± 0.22
9	0.04	2.30 ± 0.02	10800	88.00 ± 0.05	3.34 ± 0.25

(*) The preliminary values given at a lower temperature in ref. (3) are affected by an error of 10% due to a wrong percentage of TT in HT molecules.

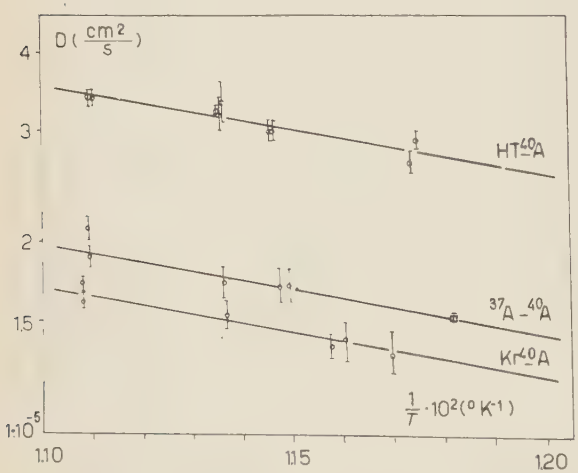

Fig. 2. – Experimental results.

length L of the capillary and the time t of diffusion. The initial concentration of the tracer in the capillary is zero. The results obtained by us are listed in Table I, II and III. The errors on D are determined taking into account those due to the concentration measurements and to the length of the capillary. No case of turbolent diffusion was found over about 20 runs. The experimental data are shown

in Fig. 2. The equations:

$$D(\text{HT-A}) = 110 \quad \exp\left[-312/T\right]\cdot 10^{-5} \text{ cm}^2/\text{s}$$

$$D(\text{A-A}) = 61 \quad \exp\left[-312/T\right]\cdot 10^{-5} \text{ cm}^2/\text{s}$$

$$D(\text{Kr-A}) = 52.5 \exp\left[-312/T\right]\cdot 10^{-5} \text{ cm}^2/\text{s}$$

are the best fit of the experimental values.

3. – Discussion of argon self-diffusion.

3·1. *Brownian motion approach.* – As we pointed out in the introduction the self-diffusion coefficient D is related to the frictional coefficient ζ, which appears in the Brownian motion approach to the transport processes in liquids, through the Einstein relationship:

$$D = \frac{kT}{\zeta} . \tag{1}$$

We shall see now firstly if the expression of ζ given by J. KIRKWOOD and other authors ([5,6]) in terms of the intermolecular pair potential $\varphi(R)$ and of the second order number density $^{\circ}n^{(2/1)}$ is correct compared to the experimental determination of ζ given by (1) through D. The expression of ζ is the following:

$$\zeta^2 = \frac{4\pi}{3} m \int_0^{\infty} \left(\frac{\partial^2 \varphi}{\partial R^2} + \frac{2}{R} \frac{\partial \varphi}{\partial R} \right) {}^{0}n^{(2/1)} R^2 \, \mathrm{d}R . \tag{2}$$

Recently the radial distribution function has been determined for liquid argon at 84° K by means of neutron diffraction ([7]).

We have calculated ζ introducing in (2) this experimental $^{\circ}n^{(2/1)}$ and using the Lennard Jones potential:

$$\varphi(R) = 4\varepsilon \left[\left(\frac{\sigma}{R} \right)^{12} - \left(\frac{\sigma}{R} \right)^{6} \right] ,$$

with $\varepsilon = 1.65 \cdot 10^{-14}$ erg and $\sigma = 3.40$ Å.

([5]) J. G. KIRKWOOD, F. P. BUFF and M. S. GREEN: *Journ. Chem. Phys.*, **17**, 988 (1949).

([6]) F. COLLINS and H. RAFFEL: *Journ. Chem. Phys.*, **29**, 699 (1958).

([7]) D. G. HENSHAW: *Phys. Rev.*, **105**, 976 (1957).

We find

$$\zeta_{\text{calc}} = 5.84 \cdot 10^{-10} \text{ g/s}$$

which can be compared with our experimental result at 84 °K

$$\zeta_{\text{exp}} = (7.70 \pm 0.10) \cdot 10^{-10} \text{ g/s} .$$

Other authors have determined by equation (2) the value of ζ with the same intermolecular potential but with different ${}^{\circ}n^{(2/1)}$ at 89 °K. With the X-ray determination of the radial distribution function ([5]) they found $\zeta_{\text{calc}} = 4.84 \cdot 10^{-10}$ g/s and with a theoretical ([8]) distribution function $\zeta_{\text{calc}} = 1.21 \cdot 10^{-10}$ g/s. These values have to be compared with our experimental value at 89 °K $\zeta_{\text{exp}} = 6.60 \cdot 10^{-10}$ g/s.

We can see therefore that in none of these cases the agreement is completely satisfactory.

Furthermore KIRKWOOD *et al.* ([5,9,10]) gave some equations for the shear viscosity η and for the thermal conductivity k of liquid argon at 89 °K in terms of the friction constant. Introducing in these equations our experimental value of ζ we find:

$$\eta_{\text{calc}} = 1.76 \cdot 10^{-3} \text{ poise} , \qquad k_{\text{calc}} = 1.77 \cdot 10^{-4} \text{ cal/g} \cdot \text{s} \cdot {}^{\circ}\text{K} ,$$

while the experimental values ([11,12]) at the same temperature, are:

$$\eta_{\text{exp}} = 2.39 \cdot 10^{-3} \text{ poise} , \qquad k_{\text{exp}} = 2.9 \cdot 10^{-4} \text{ cal/g} \cdot \text{s} \cdot {}^{\circ}\text{K} .$$

Although the agreement is not fully satisfactory, anyhow the experimental value of ζ leads to some minor improvement in the above calculated coefficients, compared with the values of k and η given in the previous papers ([5,9]).

We wish to mention however that in this theory the temperature dependence of the transport coefficients has not yet been explored.

([8]) J. G. KIRKWOOD, V. A. LEWINSON and B. J. ALDER: *Journ. Chem. Phys.*, **20**, 929 (1952).

([9]) R. W. ZWANZIG, J. G. KIRKWOOD, I. OPPENHEIM and B. J. ALDER: *Journ. Chem. Phys.*, **22**, 783 (1954).

([10]) R. W. ZWANZIG, J. G. KIRKWOOD, K. F. STRIPP and I. OPPENHEIM: *Journ. Chem. Phys.*, **21**, 2050 (1953).

([11]) N. S. RUDENKO and L. W. SCHUBNIKOW: *Phys. Zeits. Sowjetunion*, **6**, 470 (1934).

([12]) A. UHLIR: *Journ. Chem. Phys.*, **20**, 463 (1952).

([13]) J. G. KIRKWOOD and S. A. RICE: *Journ. Chem. Phys.*, **31**, 901 (1959).

Recently other authors ([6,13]) have improved the Brownian theory of transport and have related the coefficients of thermal conductivity and shear viscosity to the diffusion coefficient and to the thermodynamic properties of the system, performing all calculations very simply. However, in this way they dont seem to get better results, concerning the absolute value of the coefficients or the temperature dependence. For example the expression ([13]) for η in terms of D, T and some thermodynamic properties of the fluid, with our experimental value of the self diffusion coefficient of argon gives the following results:

T (°K)	$\eta_{calc} \cdot 10^3$ (P)	$\eta_{exp} \cdot 10^3$ (P)
84	0.9	2.80
89	1.0	2.39

From our data we can easily deduce an expression of ζ in function of temperature, for liquid Argon at a pressure of 2 atm. This expression within experimental errors is:

$$\zeta = 2.26 \cdot 10^{-13} T \exp [312/T] \text{ g/s} . \tag{3}$$

We can incidentally notice that the reduced value of ζ is:

$$\zeta^* = \zeta \frac{\sigma}{\sqrt{m\varepsilon}} = 7.3 \cdot 10^{-3} T \exp [312/T] ,$$

from which we can deduce ζ for any liquid obeying to the law of corresponding states.

3·2. *Stokes-Einstein relationship.* – An outstanding question is the relation between D and η . Even the simpler theories, suggest an intimate connection between them, as we can see for example in the hard sphere treatment of transport processes ([14,15]) and in the Eyring theory ([16-18]). This connection has

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([14]) F. COLLINS and H. RAFFAEL: *Journ. Chem. Phys.*, **22**, 1728 (1954); **23**, 1454 (1955).

([15]) H. C. LONGUET-HIGGINS and J. A. POPLE: *Journ. Chem. Phys.*, **25**, 884 (1956).

([16]) S. GLASSTONE, K. J. LEIDLER and H. EYRING: *Theory of Rate Processes* (New York, 1941), pp. 480-516.

([17]) J. C. LI and P. CHANG: *Journ. Chem. Phys.*, **23**, 518 (1955).

([18]) E. MCLOUGHLIN: *Trans. Far. Soc.*, **55**, 28 (1959).

always been expressed in a form of the type of the Stokes-Einstein relation. Therefore, disposing of a complete set of experimental values for D and η in a certain range of temperature, it can be useful to check the validity of this relation. The apparent molecular radius a as calculated from

$$\frac{kT}{D\eta} = 6\pi a \tag{4}$$

is constant within experimental errors, as shown in Table IV. The radius $a = 1.47$ Å can be compared with the value 1.5 Å, which represents the half

TABLE IV.

T (°K)	D ($\cdot 10^5$ cm^2/s)	η ($\cdot 10^3$ P)	a (Å)
84	1.49	2.81	1.466
86	1.62	2.64	1.470
88	1.77	2.46	1.476
90	1.92	2.32	1.476

nearest distance of approach of two atoms in liquid argon, measured at about 84 °K by neutron diffraction (7).

It seems, therefore, that the Stokes-Einstein relationship is valid with good accuracy also for diffusing molecules having the same size of the solvent.

4. – Discussion of binary diffusion.

4·1. – The problem of binary diffusion has not yet been fully investigated under a rigorous theoretical point of view. A phenomenological treatment (1) was given by us, applying the corresponding state principle to non equilibrium phenomena (19) in simple liquids (with Lennard-Jones intermolecular potential [6-12]) and assuming that the diffusion process follows Arrhenius' law.

We gave an expression for the diffusion coefficient of tracers in terms of the temperature and of the molecular parameters of the diffusing molecules at constant pressure. This expression is the following:

$$D = 8.7 \cdot 10^3 \frac{\sigma_1^2}{\sigma_{i1}} \sqrt{\frac{\varepsilon_{i1}}{m_{i1}}} \exp\left[-\frac{2.6\,\varepsilon_1}{kT}\right] \text{cm}^2/\text{s}, \tag{5}$$

(19) G. CINI-CASTAGNOLI, G. PIZZELLA and F. P. RICCI: *Nuovo Cimento*, **11**, 466 (1959).

where subscripts i and 1 refer to the tracer and to the abundant liquid respectively and where

$$\varepsilon_{i1} = \sqrt{\varepsilon_i \varepsilon_1} \; ; \qquad m_{i1} = \frac{2 m_i m_1}{m_i + m_1} \; ; \qquad \sigma_{i1} = \frac{\sigma_i + \sigma_1}{2} \; .$$

The values of ε and σ assumed are the following:

Substance	ε/k (°K)	σ (Å)
Argon	120	3.41
Nitrogen	100	3.70
Kripton	166	3.65
HT	37	2.92

In Table V the experimental values of D in simple liquids are listed together with those calculated by means of (5). It is worth-while to notice that these values agree within better than 6%, (which is of the same order of the experimental error).

The validity of (5) for such a great variety of substances induces us to investigate the physical meaning of eq. (5).

Eq. (5) is clearly similar to the well-known random walk expression: $D = a^2 \nu P$. The probability P of the diffusion process is here represented by the exponential term, the distance that the molecule of type i must travel in each elementary step of diffusion a is determined by σ_1, and the frequency ν of vibration of this molecule by $(1/\sigma_{i1})\sqrt{\varepsilon_{i1}/m_{i1}}$. We can therefore infer that only one type of mechanism occurs in diffusion against other suggestions ([20]) and that the probability of the elementary process is determined only by the abundant liquid, for instance through density fluctuations. Moreover it is easy to see that the vibration of the diffusing molecule is the same as if it was coupled only to one molecule of type 1 with coupling constant proportional to $\sqrt{\varepsilon_{i1}}/\sigma_{i1}$.

The contrast between the co-operative character of the exponential term, involving a great number of molecules, and the pair character of the frequency factor (mainly the presence of the reduced mass) seems to be the essential feature of expression (5). It should be interesting to see if a model of exchange based on the « two oscillator » mechanism would give the exact numerical factors appearing in eq. (5).

([20]) B. N. BROCKHOUSE: *Suppl. Nuovo Cimento*, **9**, 45 (1958).

TABLE V.

	T (°K)	D_{exp} ($\cdot 10^5$ cm^2/s)	D_{calc} ($\cdot 10^5$ cm^2/s)	$\frac{D_{exp} - D_{calc}}{D_{calc}}$ %
Argon in Nitrogen	75.62	2.41	2.30	+ 4.6
	74.74	2.20 (*)	2.21	— 0.4
	72.52	2.26	1.98	—14.0
	71.47	2.10	1.89	+11.0
	70.18	1.75	1.78	— 1.7
	68.94	1.59	1.65	— 3.7
	66.98	1.57	1.48	+ 6.0
HT in Nitrogen	72.82	3.69	3.50	+ 5.4
	71.20	3.40	3.24	+ 5.0
	70.00	3.16	3.02	+ 4.6
	67.40	2.81	2.63	+ 6.8
Kr in Argon	90.20	1.72	1.70	+ 1.2
	90.20	1.62	1.70	— 4.7
	87.94	1.54	1.55	— 0.7
	86.38	1.38	1.47	— 6.1
	86.10	1.40	1.44	— 2.8
	85.44	1.35	1.41	— 4.2
^{37}A in Argon	90.10	2.10	1.90	+10.5
	90.10	1.89	1.90	— 0.5
	87.98	1.73	1.73	0
	87.10	1.71 (*)	1.68	+ 1.8
	84.56	1.53 (**)	1.51	+ 1.3
HT in Argon	90.10	3.37 (*)	3.58	— 5.9
	88.01	3.23 (*)	3.26	— 1.0
	88.00	3.34	3.26	+ 2.5
	87.25	3.02 (*)	3.18	— 5.0
	85.17	2.69	2.93	— 8.2
	85.12	2.91	2.93	— 0.7

(*) Two runs.
(**) Five runs, see ref. (4).

4·2. – Finally calculating the apparent radius of HT and Kr in liquid argon by the relation (4), we find that they are constant within 1% in the range (84 ÷ 90) °K.

The values are $a_{HT} = 0.82$ Å and $a_{Kr} = 1.70$ Å. The radius of HT in liquid argon is in good agreement with the radius that we found in liquid nitrogen of 0.8 Å.

It is interesting to notice that within experimental errors argon and kripton have the same ratio $\sigma/a = 2.3$. This ratio is instead much higher for HT molecules. The large deviation could be due to quantum effects, although they should not be so important at this temperature.

RIASSUNTO

La diffusione di traccianti in argon liquido è stata misurata in un grande campo di temperatura e sotto una pressione di 2 atm. I risultati sono espressi dalle seguenti equazioni:

$$D(\text{HT-A}) = 110 \quad \exp[-312/T]\cdot 10^{-5}\ \text{cm}^2/\text{s}\ ,$$

$$D(^{37}\text{A-A}) = 61 \quad \exp[-312/T]\cdot 10^{-5}\ \text{cm}^2/\text{s}\ ,$$

$$D(\text{Kr-A}) = 52.5 \exp[-312/T]\cdot 10^{-5}\ \text{cm}^2/\text{s}\ .$$

I risultati ottenuti con la teoria del « Brownian motion approach » estesa ai fenomeni di trasporto sono paragonati coi dati sperimentali dell'autodiffusione. Le diffusioni binarie sono invece discusse sulla base di trattazioni fenomenologiche.

Single Particle Singularities in Scattering and Production Amplitudes.

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(ricevuto il 30 Dicembre 1959)

Summary. — The conjecture that residues at poles in two particle scattering amplitudes are correctly given by lowest order renormalized perturbation theory is proved rigorously for certain important cases. Some of the processes considered have poles in the complex momentum transfer plane lying in otherwise analytic regions. This justifies the use of extrapolation procedures for determining coupling constants and parities for these processes.

1. – Introduction.

We attempt to prove in this paper certain analyticity properties of scattering and production amplitudes regarded as functions of momentum transfer for fixed total centre-of-mass energy (1). Such properties are closely associated with single particle intermediate states. These properties are of importance in that they allow a determination of certain elementary particle coupling constants (2) and parities (3) by an extrapolation of angular distribution data

(*) This work is supported in part by the Air Research and Development Command, United States Air Force.

(1) A preliminary report of these results was presented at the Ninth International Conference on High Energy Physics (Kiev, 1959).

(2) G. F. Chew: *Phys. Rev.*, **112**, 1380 (1958); M. J. Moravcik, J. G. Taylor and J. L. Uretsky: *Phys. Rev.*, **113**, 689 (1959); G. F. Chew: *Report at the High Energy Physics Conference* (Kiev, 1959).

(3) J. G. Taylor: *Nucl. Phys.*, **9**, 357 (1959); *Phys. Rev.*, **116**, 768 (1959); M. J. Moravcik: *Phys. Rev. Lett.*, **2**, 352 (1959); S. Barshay and S. Glashow: *Phys. Rev. Lett.*, **2**, 371 (1959).

at a given centre-of-mass energy beyond the physical region to a specified (imaginary) scattering or production angle θ. Such a method has only given reliable results so far in the determination of the pion-nucleon coupling constant ([2]), though this limitation is due to a scarcity of strange particle scattering and production data.

The particular properties we wish to prove are concerned with that part of the amplitude which contains a single particle intermediate state. We wish first to prove that this single particle contribution is what we expect it to be, namely the Born approximation term as obtained in lowest order perturbation theory but written in terms of renormalized masses and coupling constants. This problem has already been considered by ZIMMERMANN ([4]) in the case that all momenta are real. We see that this will not be the case here. For example we expect our formulae to contain the vertex function $\langle 0 | j(0) | p_1 p_2 \rangle$ evaluated for $p_1^2 = p_2^2 = (p_1 + p_2)^2 = m^2$ in the case of identical neutral scalar bosons described by the current j and with mass m. This will require at least one of the momenta p_1 and p_2 to be complex. Thus a non-trivial problem of analytic continuation has to be solved. Our discussion of this problem is given in Section 2. Our proofs in this paper will be based only on the general properties of quantum field theory as used in proving the fixed-momentum transfer dispersion relations ([5]). We refer the reader to the papers listed in references ([5-8]) for a detailed discussion of these properties, and only note here that they include Lorentz invariance, local commutativity and the asymptotic condition. We would like to emphasize that no specific form of interaction Hamiltonian or field equation is used in our discussion.

We discuss in Section **3** the relation between the region of analyticity in $\cos\theta$ of the remainder of the amplitude (without the single particle contribution) and the position of the pole in the single particle contribution. In particular we determine possible values of the total energy for which the single particle pole lies inside the analyticity region of the remainder.

We conclude with a brief discussion of our results in Section **4**.

([4]) W. ZIMMERMANN: *Nuovo Cimento*, **13**, 503 (1959).

([5]) N. N. BOGOLJUBOV, B. MEDVEDEV and M. POLIVANOV: *Lecture Notes at Institute for Advanced Study* (Princeton, 1956); H. J. BREMERMANN, R. OEHME and J. G. TAYLOR: *Phys. Rev.*, **109**, 2178 (1958).

([6]) R. HAAG: *Kgl. Dan. Vid. Selsk., Mat.-fys. Medd.*, **29**, no. 12 (1955); A. S. WIGHTMANN: *Proc. of the « Colloque sur les Problèmes Mathématiques de la Théorie Quantique des Champs »* (Lille, 1957); W. SCHMIDT and K. BAUMANN: *Nuovo Cimento*, **4**, 860 (1956).

([7]) J. G. TAYLOR: *Lectures on Dispersion Relations in Quantum Field Theory*, vol. **1** University of Maryland (July 1958).

([8]) H. LEHMANN, K. SYMANZIK and W. ZIMMERMANN: *Nuovo Cimento*, **1**, 205 (1955).

2. – Single particle singularities.

We consider here the process of elastic scattering of two neutral scalar bosons in the case where all four external masses may be different. The initial an final momenta will be denoted by p_1, p_2 and p_3, p_4 respectively, with corresponding masses m_1, ..., m_4.

Then by the usual reduction formulae (6) the scattering amplitude $M(p_1p_2;\, p_3p_4)$ is

$$(1)\qquad M(p_1p_2p_3p_4) = i(4p_{10}p_{20})^{\frac{1}{2}}\int \exp\,[-ip_3x_3]\theta(x_{30})\,\langle 0\,|\,[j_3(x_3),\, j_4(0)]\,|\,p_1p_2\rangle\,\mathrm{d}x_3\,,$$

with absorptive part

$$(2)\qquad A(p_1p_2p_3p_4) = \int \exp\,[-ip_3x_3 + ip_1x_1 + ip_2x_2]\cdot$$

$$\cdot\langle 0\,|\,\frac{\delta^2}{\delta\varphi_1(x_1)\,\delta\varphi_2(x_2)}\,[j_3(x_3),\, j_4(0)]\,|\,0\rangle\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,.$$

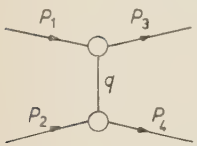

Fig. 1. – Diagram for process giving singular contribution to the amplitude $M(p_1p_2;\, p_3p_4)$ of eq. (1) with a singularity at $(p_1 - p_3)^2 = m^2$.

In equation (2) the functional derivative $\delta/\delta\varphi$ of BOGOLJUBOV (5) is used for convenience, whilst j_l is the current for the boson of type l. The contribution to $M(p_1p_2p_3p_4)$ with a singularity at $(p_1 - p_3)^2 = m^2$, where m is the mass of the possible intermediate boson of Fig. 1, is seen from equation (2) to come from the term involving a single boson intermediate state in the expression

$$\int \exp\,[-ip_3x_3 + ip_1x_1 + ip_2x_2]\,\langle 0\,|\left[\frac{\delta}{\delta\varphi_1(x_1)}\,j_3(x_3),\, \frac{\delta}{\delta\varphi_2(x_2)}\,j_4(0)\right]|\,0\rangle\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,.$$

That part of $A(p_1p_2p_3p_4)$ giving rise to such a singularity is equal to

$$(3)\qquad A_{1\pi}(p_1p_2p_3p_4) = \int \exp\,[-ip_3x_3]\,\mathrm{d}x_3 \int \mathrm{d}q\,\theta(q_0)\,\delta(q^2 - m^2)\,[\langle 0\,|\,j_3(x_3)\,|\,p_1q\rangle\cdot$$

$$\cdot\langle q\,|\,j_4(0)\,|\,p_2\rangle - \langle 0\,|\,j_4(0)\,|\,p_2q\rangle\langle q\,|\,j_3(x)\,|\,p_1\rangle] =$$

$$= \int \exp\,[i(p_1 - p_3)x]\,\mathrm{d}x \int \exp\,[iqx]\,F(q)\,\mathrm{d}q\,.$$

where

$$F(q) = \delta(q^2 - m^2)\,\varepsilon(q_0)\langle 0\,|\,j_3(0)\,|\,p_1q\rangle\langle q\,|\,j_4(0)\,|\,p_2\rangle\,.$$

The interchange of q and x integration in equation (3) is not immediately permitted, since at least one of the two vectors p_1, p_3 which satisfy $p_1^2 = m_1^2$, $p_3^2 = m_3^2$, $(p_1 - p_3)^2 = m^2$ may be complex. This will certainly be so for the actual processes considered at the end of this section. We wish to find conditions under which this interchange of orders of integration will be permitted.

From Lorentz invariance $A_{1\pi}(p_1 p_2 p_3 p_4)$ is a function only of six independent scalar products which may be formed from three independent vectors of the set p_1, p_2, p_3, p_4. We choose particular values for these vectors so that $(p_1 - p_3)$ is a real vector, $(p_1 + p_2)^2 = W$, $W > (m_1 + m_2)^2$, and $(p_1 - p_3)^2 = \Delta^2$ with Δ^2 taking any real value, by

$$
(4) \qquad \left| \begin{array}{l}
p_1 = (\omega, \sqrt{\omega^2 - m_1^2}\, \boldsymbol{e}_1) \,, \\
p_3 - p_1 = (\sqrt{\Delta^2 + \beta},\ \beta^{\frac{1}{2}} \boldsymbol{e}_3) \,, \\
p_2 = (p_{20}, \sqrt{p_{20}^2 - m_2^2}\, \boldsymbol{e}_2) \,, \\
p_{20} = \dfrac{W - m_1^2 - m_2^2}{2\omega} \,, \qquad \omega = \dfrac{m_3^2 - m_1^2 - \Delta^2}{2\sqrt{\Delta^2 + \beta}} \,, \\
\beta \geqslant \max\left\{0,\ \dfrac{(m_3^2 - m_1^2 - \Delta^2)^2}{4 m_1^2} - \Delta^2\right\} \,, \\
\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = \boldsymbol{e}_1 \cdot \boldsymbol{e}_3 = \boldsymbol{e}_2 \cdot \boldsymbol{e}_3 = 0 \,, \qquad \boldsymbol{e}_1^2 = \boldsymbol{e}_2^2 = \boldsymbol{e}_3^2 = 1 \,.
\end{array} \right.
$$

We note that $p_1^2 \neq m_1^2$ if equation (4) is satisfied. Further p_2 is a real vector for all real values of Δ^2, whilst $\omega^2 < m_1^2$. Thus

$$
\operatorname{Im} p_1 q = \boldsymbol{e}_1 \cdot \boldsymbol{q} \sqrt{m_1^2 - \omega^2} \,, \qquad \operatorname{Re} p_1 q = \omega q_0 \,.
$$

We introduce

$$
F_3((p_1 + q)^2) = \langle 0 \,|\, j_3(0) \,|\, p_1 q \rangle \,,
$$

$$
F_4((p_2 - q)^2) = \langle q \,|\, j_4(0) \,|\, p_2 \rangle \,.
$$

Then we may write

$$
(5) \qquad F(q) = \varepsilon(q_0)\, \delta(q^2 - m^2)\, F_3(m^2 + m_1^2 + 2 p_1 q)\, F_4(m_2^2 + m^2 - 2 p_2 q) \,.
$$

We wish to find conditions on $F_3(z)$, $F_4(z)$ so that $F(q)$ defined by the right-hand side of equation (5) is a tempered distribution in q, so allowing the integrations over q and x in equation (3) to be interchanged. We assert that a sufficient condition is that for $i = 3$ and 4, $F_i(z)$ is analytic in the cut z-plane, with a cut starting to the right of m_i^2 and extending to $+\infty$; we denote this condition by (A_i). Let us assume that (A_3) and $(A\ )$ are satisfied and use this to show that $F(q)$ is tempered; we will consider the validity of (A_3) and (A_4) for certain interesting processes at the end of this section.

We split up the region of integration into $q_0 > m$ and $q_0 < -m$. For $q_0 > m$ and $p_{20} > 0$ (or $\Delta^2 < m_3^2 - m_1^2$) then $m_2^2 + m^2 - 2p_2q < (m_2 - m)^2 < m_4^2$ for stable particles, so $F_4(m_2^2 + m^2 - 2p_2q)$ is an analytic function of q. Since $\delta(q^2 - m^2) \cdot F_3(m^2 + m_1^2 + 2p_1q)$ is then a tempered distribution we have that $F(q)$ is also. For $q_0 > m$ and $p_{20} > m$ and $p_{20} < 0$ (or $\Delta^2 > m_3^2 - m_1^2$) then $\delta(q^2 - m^2)F_4(m^2 + m_2^2 - 2p_2q)$ is tempered in q and $F_3(m^2 + m_1^2 + 2p_1q)$ is an analytic function of q for $\boldsymbol{q} \cdot \boldsymbol{e}_- \neq 0$ or $q_0 > (m^2 + m_1^2 - m_3^2)/2m_1$. Thus $F(q)$ may be defined as a tempered distribution in q on the space of test functions which vanish if $\boldsymbol{q} \cdot \boldsymbol{e}_1 = 0$ or if $q_0 < (m^2 + m_1^2 - m_3^2)/2m_1$. We can extend [9] F to a tempered distribution $\bar{F}$ so that $\bar{F} = F$ if $\boldsymbol{q} \cdot \boldsymbol{e}_1 \neq 0$ or if $q_0 > (m^2 + m_1^2 - m_3^2)/2m_1$. Thus for $q_0 > m$ we have shown that $F(q)$ may be defined as a tempered distribution. We may again extend $\bar{F}(q)$ to be a tempered distribution defined on the space of all test functions.

Thus we have shown that $F(q)$ may be defined by the right-hand side of equation (5) or a suitable extension of it. Then we may interchange the orders of integration in equation (3) to obtain

$$A_{1\pi}(W, \Delta^2, p_4^2) = F(p_3 - p_1) = \delta(\Delta^2 - m^2) F_3(m_3^2) F_4(p_4^2) , \tag{6}$$

where

$$p_4^2 = m^2 + m_2^2 + \frac{2(\beta + m^2)(W - m_1^2 - m_2^2)}{m_1^2 + m^2 - m_4^2} .$$

We note that at $q = p_3 - p_1$, $\Delta^2 = m^2$ then $q_0 > m$ and $q_0 > (m^2 + m_1^2 - m_3^2)/2m_1$ for stable particles if $m^2 + m_1 > m_3^2$. Thus the extension of F described above does not enter equation (6). The quantity $F_4(p_4^2)$ is evaluated on the upper side of the cut, since it is the retarded function. If we now fix W in equation (6), and since Δ^2 is fixed at m^2, then $A_{1\pi}$ regarded as a function of p_4^2, is the boundary value on the upper cut of a function analytic in a cut plane with a cut from the right of m_4^2 to ∞. We may continue $A_{1\pi}$, in a unique fashion to $p_4^2 = m_4^2$ and finally obtain the desired result,

$$A_{1\pi}(W, \Delta^2) = \delta(\Delta^2 - m^2) F_3(m_3^2) F_4(m_4^2) . \tag{7}$$

We repeat that this result is valid provided conditions (A_3) and (A_4) are satisfied, or in other words that the corresponding vertex functions F_3 and F_4 satisfy dispersion relations. A more powerful result holds if $m_1^2 + m^2 < m_3^2$ or $m_3^2 + m^2 < m_1^2$, this being that equation (7) is valid if condition (A_4) holds, as may be seen from the above method of proof. Some condition must be

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[9] N. BOURBAKI: *Eléments de Mathématique*, **5**, 74, cor. 5 (A.S.I. 1189).

imposed on $F_3(z)$ to make equation (7) sensible, say that F_3 is analytic at $z = m_3^2$. In the particular cases we consider in Table I, if condition (A_3) is not satisfied and $m_1^2 + m^2 < m_3^2$ or $m_3^2 + m^2 < m_1^2$, F_3 cannot be shown to be analytic at m_3^2. For this reason we will not consider this extension any further here, but simply use that if both (A_3) and (A_4) are satisfied then the single particle absorptive part is as given by equation (7).

TABLE I. – *Table showing the vertex functions wich arise in the one particle singularities for various processes.* In column 2 for the singularity in $(p_1 - p_3)^2$ and in column 5 for that in $(p_1 - p_4)^2$. Columns 3 and 6 give the region of analyticity of the vertex function on the same row and preceding column, and columns 4 and 7 give the value to which the invariant momentum p^2 in eqs. (6) and (8) must be continued.

Process	Vertex functions	Position of cut	Physical value of p^2	Vertex functions	Position of cut	Physical value of p^2
1	2	3	4	5	6	7
1. $\mathcal{N} + \mathcal{N} \to \mathcal{N} + \mathcal{N}$	$(\mathcal{N}\mathcal{N}\pi)$	$(M_{\mathcal{N}} + M_\pi)^2$	$M_{\mathcal{N}}^2$	$(\mathcal{N}\mathcal{N}\pi)$	$(M_{\mathcal{N}} + M_\pi)^2$	$M_{\mathcal{N}}^2$
2. $\gamma + \mathcal{N} \to \pi + \mathcal{N}$	$(\gamma\pi\pi)$	$(M_\pi + M_\gamma)^2$ (1)	M_π^2	$(\mathcal{N}\gamma\mathcal{N})$	Not analytic in cut plane	M_γ^2
	$(\mathcal{N}\mathcal{N}\pi)$	$(M_{\mathcal{N}} + M_\pi)^2$	$M_{\mathcal{N}}^2$	$(\mathcal{N}\mathcal{N}\pi)$	$(M_{\mathcal{N}} + M_\pi)^2$	$M_{\mathcal{N}}^2$
3. $\gamma + \mathcal{N} \to \mathrm{K} + \mathrm{Y}$ (3)	$(\gamma\mathrm{K}\mathrm{K})$	$(M_\mathrm{K} + M_\gamma)^2$ (1)	M_K^2	$(\gamma\mathrm{Y}\mathrm{Y})$	$(M_\mathrm{Y} + M_\gamma)^2$ (1)	M_Y^2
	$(\mathrm{K}\mathrm{Y}\mathcal{N})$	$(M_\mathrm{K} + M_{\mathcal{N}})^2$ (2)	M_Y^2	$(\mathcal{N}\mathrm{K}\mathrm{Y})$	$(M_{\mathcal{N}} + M_\mathrm{Y})^2$ (2)	M_K^2
4. $\mathrm{K} + \mathcal{N} \to \mathrm{K} + \mathcal{N}$	$(\pi\mathrm{K}\mathrm{K})$	$(M_\mathrm{K} + M_\pi)^2$	M_K^2	$(\mathcal{N}\mathrm{K}\mathrm{Y})$	$(M_{\mathcal{N}} + M_\mathrm{Y})^2$ (2)	M_K^2
	$(\mathcal{N}\mathcal{N}\pi)$	$(M_{\mathcal{N}} + M_\pi)^2$	$M_{\mathcal{N}}^2$	$(\mathrm{K}\mathcal{N}\mathrm{Y})$	Not analytic in cut plane	$M_{\mathcal{N}}^2$
5. $\mathrm{K} + \mathcal{N} \to \pi + \mathrm{Y}$ (3)	$(\pi\mathrm{K}\mathrm{K})$	$(M_\mathrm{K} + M_\pi)^2$	M_K^2	$(\pi\mathcal{N}\mathcal{N})$	$(M_{\mathcal{N}} + M)_\pi{}^2$	$M_{\mathcal{N}}^2$
	$(\mathrm{K}\mathcal{N}\mathrm{Y})$	Not analytic in cut plane	$M_{\mathcal{N}}^2$	$(\mathcal{N}\mathrm{K}\mathrm{Y})$	$(M_{\mathcal{N}} + M_\mathrm{Y})^2$ (2)	M_K^2
6. $\pi + \mathcal{N} \to \mathrm{K} + \mathrm{Y}$	$(\pi\mathrm{K}\mathrm{K})$	$(M_\mathrm{K} + M_\pi)^2$	M_K^2	$(\pi\mathrm{Y}\mathrm{Y})$	$(M_\mathrm{Y} + M_\pi)^2$	M_Y^2
	$(\mathrm{K}\mathrm{Y}\mathcal{N})$	$(M_\mathrm{K} + M_{\mathcal{N}})^2$ (2)	M_Y^2	$(\mathcal{N}\mathrm{K}\mathrm{Y})$	$(M_{\mathcal{N}} + M_\mathrm{Y})^2$ (2)	M_K^2

(1) M_γ is the small photon mass which is finally to be taken zero. The analyticity is correct to all orders in the electromagnetic charge e; all other results in the table are only valid to lowest order in e.

(2) Valid in the approximation that the lowest mass intermediate state coupled to a K-meson is a baryon-antibaryon pair. This corresponds to neglecting any direct K-π interactions in comparison with K-baryon interactions.

(3) Y denotes a Λ or Σ hyperon.

An alternative method to prove that (A_3) and (A_4) are sufficient conditions is to continue directly in the invariants, without using possibly unfamiliar tools of functional analysis. We give both these methods here because either

method may be of importance in extending analyticity properties more generally. We choose a co-ordinate system for which

$$p_1 = (m_1, 0), \quad p_2 = (p_{20}, \sqrt{p_{20}^2 - m_2^2}\, \boldsymbol{e}), \quad p_3 = (\omega, \sqrt{\omega^2 - \beta}\, \boldsymbol{f}).$$

The scattering amplitude of equation (1) is a function $(M(W, \Delta^2, \beta, \zeta)$ of the six invariants $p_1^2 = m_1^2$, $p_2^2 = m_2^2$, $p_3^2 = \beta$, $(p_1 - p_3)^2 = \Delta^2$, $(p_1 + p)^2 = W$, $(p_1 + p_2 - p_3)^2 = \zeta$. Then from equation (3) we obtain $A_{1\pi}$ as a function of W, Δ^2, β, ζ to be

$$A_{1\pi}(W, \Delta^2, \beta, \zeta) = \varepsilon(p_{30} - p_{10})\, \delta(\Delta^2 - m^2) F_3(\beta) F_4(\zeta). \tag{8}$$

We note that for $\beta < -m_1^2$ the vectors $p_1, \ldots, p_1$ entering into equation (3) are real so that the interchange of orders of x and q integration is justified. If we now continue the right-hand side of equation (8) onto the mass shell $\beta = m_3^2$, $\zeta = m_4^2$ we obtain again equation (7). This continuation can be performed if conditions (A_3) and (A_4) are satisfied. For β sufficiently negative, ζ always lies on the cut, so care has to be taken to continue $F_4(\zeta)$ on the correct side of the cut. The relation between ζ and β is

$$\zeta = m^2 + m_2^2 + \frac{p_{20}}{m_1}(m^2 + m_1^2) - \frac{p_{20}}{m_1}\beta + \frac{\boldsymbol{e}\cdot\boldsymbol{f}}{m_1}(p_{20}^2 - m_2^2)^{\frac{1}{2}}[\beta - (m_1 + m_2)^2]^{\frac{1}{2}}[\beta - (m_1 - m_2)^2]^{\frac{1}{2}}. \tag{9}$$

Keeping $(p_1 + p_2)^2$, $(p_1 - p_3)^2$ fixed at the values W^2, m^2, we are left with the two independent parameters β and $\boldsymbol{e}\cdot\boldsymbol{f}$ arbitrarily variable. It is always possible to vary these in such a manner that continuation of both $F_3(\beta)$ and $F_4(\zeta)$ to $\beta = m_3^2$ and $\zeta = m_4^2$ is obtained. Suitable paths are shown in Fig. 2. Allowing β to assume a small negative imaginary part, as along path a, then ζ moves into the upper half plane when $\boldsymbol{e}\cdot\boldsymbol{f}$ is real and of absolute value less than unity, while $\mathrm{Re}\,\beta$ is sufficiently negative. The branch points of ζ as a function of β can be avoided by taking $\boldsymbol{e}\cdot\boldsymbol{f} = 0$ while continuing in β.

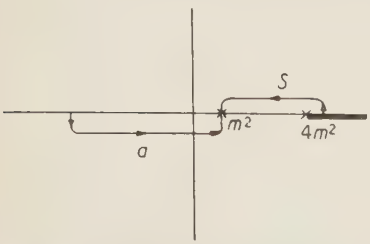

Fig. 2. Possible contours for β and ζ to continue to the physical values m_3^2 and m_4^2, when $\boldsymbol{e}\cdot\boldsymbol{f} = 0$ in eq. (9).

Our previous discussion for scalar particles may be extended immediately to particles with arbitrary spin. The amplitude for such a process will be a matrix in spin and isotopic spin space. It may be given as an expansion with

respect to a complete set of irreducible forms in the spin and isotopic spin space. The coefficients of these forms are invariant functions of the momenta only, so will have the same singularities as for the scalar particle case. It is of course necessary to observe in this case all selection rules which are relevant for the spectral conditions and which follow from conservation of isotopic spin or other quantities which do not exist for particles with zero spin and isotopic spin.

We now consider six processes for which poles have been conjectured and tentative extrapolations to poles have been performed ([2,3]). The processes are shown in the first column of Table I. In the second column are shown the vertex functions which arise in the single particle singularity as given by equation (7). The third column gives the position of the cut in the complex p^2 plane for the particular vertex function in the same row in column 2 and the fourth column gives the physical value of p^2 which is to be continued to. The analyticity region for the vertex functions is obtained by applying the Jost-Lehmann-Dyson representation ([10]) to the absorptive part of the vertex function, regarded as a function of an external mass τ other than p^2. The same method as in reference ([5]) is then used to continue the dispersion relation for the vertex function in p^2, valid initially for negative values of τ, to the actual physical mass τ. The singularities described by column 2 correspond to ascribing the momenta p_1 and p_3 to the bosons and p_2, p_4 to the fermions. There is also a one particle singularity in $(p_1-p_4)^2$. The vertex functions which occur for this singularity are given in column 5 and the position of the cut and the physical value in the p^2 plane are given in columns 6 and 7. We note that if θ is the angle between $\boldsymbol{p}_1$ and $\boldsymbol{p}_3$ in the centre of mass of the incoming system the singularity in $(p_1-p_3)^2$ occurs for values of $\cos\theta > 1$ whilst that in $(p_1-p_4)^2$ occurs for values of $\cos\theta < -1$.

In order to extend our discussion to processes with more than two initial or final particles it is necessary to discuss the analyticity properties of amplitudes involving at least two initial and two final particles. This introduces certain complications which we shall not discuss here.

3. – Analyticity properties.

We now wish to discuss analyticity properties in the invariant momentum transfer Δ^2 of the complete scattering amplitude which follows from the previous section. Let us consider explicitly the case of the elastic scattering of two identical neutral scalar mesons of mass m, using the same notation as

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([10]) R. Jost and H. Lehmann: *Nuovo Cimento*, **5**, 1598 (1957); F. J. Dyson: *Phys. Rev.*, **110**, 1450 (1958). See also vol. 2 of ref. ([7]).

in the previous section. We denote by Δ^2 the invariant momentum transfer $(p_1 - p_3)^2$ and by $N(p_1p_2; p_3p_4)$ the function $(\Delta^2 - m^2)M(p_1p_2; p_3p_4)$, where M is given by equation (1). Then from this equation

$$N(p_1p_2p_3p_4) = \int \exp[-ip_3x]\,\theta(x_0)\,F(x)\,\mathrm{d}^4x\,, \tag{10}$$

where the Fourier transform of $F(x)$ is $\bar{F}(p_3)$ equal to

$$\bar{F}(p_3) = (\Delta^2 - m^2)A(p_1p_2p_3p_4)\,,$$

so that $\bar{F}(p_3)$ has no contribution from the single particle term $A_{1\pi}(p_1p_2p_3p_4)$ since this was shown in the previous section to contain a factor $\delta(\Delta^2 - m^2)$. We may apply the Jost-Lehmann-Dyson [10] representation directly to continue N, regarded as a function of $\cos\theta$, the cosine of the centre of mass angle θ for fixed total energy $W^2 = (p_1 + p_2)^2$. N is then analytic inside an ellipse in the complex $\cos\theta$-plane with centre the origin and major semi-axis $(1 + m^4/k^2W^2)$ along the real $\cos\theta$-axis, k being the centre of mass momentum. In equation (10) there is actually a further contribution to the right-hand side coming from the equal-time commutator term

$$\int \exp[-ip_3x][-2p_{10}\,\delta(x_{30}) + \delta(x_{30})\,\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\,\delta(x_{30})]\langle 0\,|\,[j_3(x_3),\, j_4(0)]\,|\,p_1p_2\rangle\,\mathrm{d}x_3\,.$$

From causality this term can only be a polynomial in the components of p_3 with coefficients which are functions of W only, and so is evidently analytic in the whole $\cos\theta$-plane. Then from the previous section we may represent $M(W, \cos\theta)$ as

$$M(W, \cos\theta) = \frac{g^2}{(\Delta^2 - m^2)} + f(W, \cos\theta)\,, \tag{11}$$

where $f(W, \cos\theta)$ is analytic inside the above ellipse, and $g = \langle 0\,|\,j(0)\,|\,p_1q\rangle$, with $p_1^2 = q^2 = (p_1 - q)^2 = m^2$. In this case the pole term always lies outside the ellipse. To prove that the pole term lies inside the region af analyticity in $\cos\theta$ of $f(W, \cos\theta)$ it will be necessary to use a more powerful analysis than is known at present.

The pole term arising from single pion exchange in nucleon-nucleon scattering also lies outside the ellipse of analyticity for the rest of the scattering amplitude. This is not the case for photo-production of pions from nucleons where the ellipse of analyticity in the $\cos\theta$-plane has semimajor axis $\{1 + 8M_\pi^3(2M_\Lambda + M_\pi)/k'^2[W^2 - (M_\Lambda - 2M_\pi)^2]\}$, with k' the final state centre of

mass momentum. This ellipse has been derived by neglecting all states coupled to the initial photon and nucleon state which contain at least one photon, *i.e.*, we neglect all powers of the charge e higher than the first. The pole term in this case is the meson current term with a pole at $\cos\theta = V_\pi^{-1}$, where V_π is the pion velocity. For photon laboratory energies up to 1 GeV this pole lies inside the analyticity ellipse, so proving up to this energy the analyticity used by MORAVCIK, TAYLOR and URETSKY [2].

A similar analysis applied to the photoproduction of K mesons on nucleons shows that for this case the meson current pole never lies inside the analyticity ellipse if the lowest mass state coupled to a K meson is a K meson plus a π meson. If intermediate states containing only K mesons and pions can be neglected relative to baryon-antibaryon pairs then the one-meson pole lies inside the new analyticity ellipse for incident photon laboratory energies up to 4.27 GeV.

We expect to find poles in otherwise analytic regions only when we have a weak enough interaction at one vertex, so that this interaction may be taken only to the first order. This holds well for the electromagnetic interaction and should be expected to hold also for the KKπ interaction proposed by several people. PAIS [11] has given reasons to expect a coupling constant of the order of the electromagnetic coupling constant. We can show, moreover in the case of K-nucleon scattering that the one pion exchange pole lies inside the ellipse of analyticity for K laboratory energies between a value just above threshold up to 2.4 GeV, where all powers in the KKπ coupling constant are considered. However the processes $\pi + \mathcal{N} \to K + \Lambda, \Sigma$ and $K + \mathcal{N} \to \pi + \Lambda, \Sigma$, have poles arising from the proposed KKπ interaction only outside their respective analyticity ellipses. These results are summarized in Table II. We do not

TABLE II. – *Range of energies for which the single particle pole in* $(p_1 - p_3)^2$ *lies inside the ellipse of analyticity for the remainder of the amplitude.*

Process	Energy for which pole lies insie analyticity ellipse
1, $\mathcal{N} + \mathcal{N} \to \mathcal{N} + \mathcal{N}$	None
2. $\gamma + \mathcal{N} \to \pi + \mathcal{N}$	Photon lab. energy < 1 GeV [1]
3. $\gamma + \mathcal{N} \to K + Y$	Photon lab. energy $< 4{\cdot}27$ GeV [2]
4. $K + \mathcal{N} \to K + \mathcal{N}$	K-meson lab. energy < 2.4 GeV
5. $K + \mathcal{N} \to \pi + Y$	None
6. $\pi + \mathcal{N} \to K + Y$	None

[1] Correct to lowest order in e.
[2] Correct assuming lowest mass state coupled to a K-meson is a baryon-antibaryon pair.

[11] A. PAIS: *Phys. Rev.*, **112**, 624 (1958).

consider here the poles arising for negative values of $\cos\theta$ *i.e.*, in $(p_1 - p_1)^2$ since they lie outside the analyticity ellipse for the processes we are considering, except for the symmetric cases of nucleon-nucleon and K meson-nucleon scattering.

4. – Conclusion.

The single particle contribution to the scattering amplitude was shown in Section 2 to have the expected Born approximation structure in terms of renormalized quantities provided the relevant pair of vertex functions is analytic in a cut plane in one of the invariants with a suitable cut. The position of the cut allows for anomalous thresholds. It is not possible to prove this cut plane analyticity for certain of the vertex functions using present methods. It seems that a less stringent requirement of analyticity in a strip on either side of the real axis, and with a cut as before, is still sufficient to continue the right-hand side of equation (8) to the physical masses. Such a weaker analyticity requirement may be easier to prove from general properties.

The range of centre-of-mass energies for which the single particle pole lies inside the region of analyticity of the remainder of the amplitude was considered in Section 3. The small range of values of this energy for which this holds shows that either we must use the general properties much more efficiently or must use more properties than we have up to now. We see that such an extension is certainly necessary to prove the analytic properties conjectured by MANDELSTAM ([12]).

* * *

One of us (J.G.) would like to thank the Department of Scientific and Industrial Research for a grant.

([12]) S. L. MANDELSTAM: *Phys. Rev.*, **112**, 1344 (1958).

RIASSUNTO (*)

Si prova che in alcuni casi importanti la congettura che i residui ai poli nelle ampiezze di scattering a due particelle siano dati correttamente nella teoria della perturbazione rinormalizzata nel più basso ordine, è esatto. Alcuni dei processi considerati hanno poli nel piano complesso del trasferimento della quantità di moto che giace in regioni peraltro analitiche. Questo giustifica l'uso di procedimenti di estrapolazione per determinare le costanti di accoppiamento e le parità di questi processi.

(*) *Traduzione a cura della Redazione.*

Radiative Corrections to Electron-Electron Scattering.

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(ricevuto il 1° Febbraio 1960)

Summary. — We have computed the e^6 corrections to the differential cross-section for Møller scattering. The divergences for high values of the momentum of the virtual photons have been removed by the usual techniques of renormalization. The infrared divergence has been eliminated by adding the cross-section for the bremsstrahlung process associated with Møller scattering. Numerical computations give raisonable results for the corrections.

1. – Introduction.

In the last years a remarkable number of theoretical investigations and experimental suggestions [1-8] has been devoted to a lowering of limits of validity of Q.E.D. The possibility of using to this end merely electrodynamic processes (*i.e.* involving electrons and photons only) on the other hand makes it necessary the calculation of the terms of higher order in α in the expansion of S matrix. The knowledge of these terms is essential for the correct interpretation of experimental data. In fact one should compare experimental results with theoretical calculations pushed to such an order in α that the contribution from higher approximations is smaller than the experimental

(1) P. Budini, G. Poiani and I. Reina: *Int. Conf. Padova*, IX, 17 (1957).
(2) S. D. Drell: *Ann. Phys.*, **4**, 75 (1958).
(3) G. Poiani and I. Reina: *Nuovo Cimento*, **13**, 19 (1959).
(4) G. Andreassi, P. Budini and I. Reina: *Nuovo Cimento*, **12**, 488 (1959).
(5) G. Furlan and G. Peressutti: *Nuovo Cimento*, **14**, 758 (1959).
(6) J. D. Bjorken, S. D. Drell and S. C. Frautschi: *Phys. Rev.*, **112**, 1409 (1958).
(7) W. K. H. Panofsky: *Ann. Int. Conf. CERN* (Geneve, 1958), p. 3.
(8) B. Richter: *Phys. Rev. Lett.*, **3**, 114 (1958).

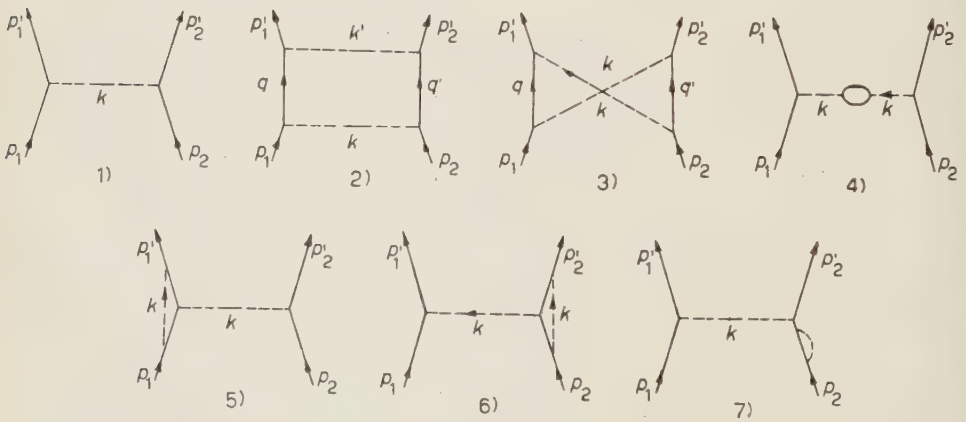
errors. Only in the case that such a comparison should show greater discrepancies than the contribution of the further terms calculated by the local theory (« radiative corrections ») one could speak of deviations from the local Q.E.D.

Radiative corrections for electrodynamic processes have been first calculated, in their classical works, by SCHWINGER ([9]) for the electron scattering in an external field and by BROWN and FEYNMAN ([10]) for the Compton effect. For $e^- - e^-$ scattering analogous calculations have been performed, as quoted in ([10]) by LOMANITZ (*) and also by AKHIEZER and POLOVIN ([11]). The latter authors however confine their considerations to the extreme relativistic case ($p \gg m$ with p momentum of the electron in C.M.) and to scattering angles, still in C.M., near to $0°$ and π^0. Now (see ([5])) the maximum non-local effect for Moller scattering occurs for scattering angles in C.M. $\simeq \pi/2$. Therefore just in this region, in view of the experiments projected *e.g.* at Stanford, we are mainly interested in the knowledge of second order corrections.

In this work we shall calculate the α^3 corrections for $e^- - e^-$ scattering cross-section. These corrections result, as well known, from the possibility of either electron emitting and reabsorbing a virtual photon or from the direct scattering with exchange of the two photons.

2. – Matrix elements.

The general situation is characterized by the following Feynman's graphs:



([9]) J. SCHWINGER: *Phys. Rev.*, **76**, 790 (1949).

([10]) L. M. BROWN and R. P. FEYNMAN: *Phys. Rev.*, **85**, 231 (1952).

(*) In spite of the kindness of Dr. LOMANITZ and of the Reference Librarian of Cornell University, we haven't yet received the above work.

([11]) AKHIEZER and POLOVIN: *Proc. Acad. Sci. USSR*, **90**, 55 (1953).

and as many graphs corresponding to exchange processes can be obtained by the substitution $p_1' \leftrightarrow p_2'$, $p_1 \leftrightarrow p_1$, $p_2 \leftrightarrow p_2$.

Moreover we must add to number 7) three more diagrams which can be obtained by inserting the virtual photon in each of the external electron lines.

The matrix elements associated with diagrams 4), 5), 6), 7) are divergent for large values of momenta of the virtual photons. These divergences can be removed by means of mass and charge renormalization and gauge invariance. As a result of this procedure the diagrams of number 7) do not give finite contribution and can be omitted. In a purely formal way ([12]) the matrix element corresponding to diagram 4) can be directly obtained from the fundamental matrix element 1) with the substitution

$$g_{\mu\nu}\frac{1}{k^2} \to g_{\mu\nu}\Pi_F(k^2)\cdot\frac{\alpha}{2\pi}\,.$$

Analogously for the diagrams 5), 6) with the substitution

$$\gamma_\mu \to \Lambda_{\mu F}\cdot\frac{\alpha}{2\pi}\,.$$

The finite quantities Π_F and $\Lambda_{\mu F}$ are

$$\Pi_F(k^2) = -\frac{1}{k^2}\frac{2}{3}\left[\frac{5}{3}-\frac{1}{3}-\left(1-\frac{1}{2\varrho}\right)\sqrt{1+\frac{1}{\varrho}}\ln\frac{\sqrt{1+1/\varrho}+1}{\sqrt{1+1/\varrho}-1}\right], \qquad \varrho = k^2/4m^2,$$

$$\Lambda_{\mu F}(p,p') = -\int_0^1 \mathrm{d}x\int_0^x \mathrm{d}y\left\{\frac{K_\mu}{a^2}+\gamma_\mu\int_0^1 \mathrm{d}z\,\frac{a^2-m^2x^2}{m^2x^2+(a^2-m^2x^2)z}-\right.$$

$$\left.-2m^2\gamma_\mu\left(1-x-\frac{1}{2}x^2\right)\frac{a^2-m^2x^2}{a^2m^2x^2}\right\} = A\gamma_\mu + B\sigma_{\mu\nu}k_\nu' + Ck_\mu'\,.$$

$$a^2 = m^2x^2 + \Lambda^2(1-x) + k'^2y(x-y)\,, \qquad k' = p - p'.$$

$$K_\mu = \gamma_\mu k'^2(1-x+y)(1-y) + ik_\mu' m(1+x)(x-2y) - \sigma_{\mu\nu}k_\nu' mx(1-x)\,.$$

The main difficulty of the work consists then in evaluating certain integrals connected to diagrams 2), 3), characteristic of the process, and which describe to the e^4 order the direct scattering.

([12]) See *e.g.* J. M. JAUCH and F. ROHRLICH: *The Theory of Photons and Electrons* (Cambridge, 1955).

The various matrix elements are finally

$$
(1)\quad
\begin{cases}
M_1 = \dfrac{e^2}{4\pi^2 i}\,\dfrac{m^2}{\sqrt{\varepsilon_1\varepsilon_2\varepsilon_1'\varepsilon_2'}}\,\dfrac{\overline{u}(p_1')\gamma_\mu u(p_1)\,\overline{u}(p_2')\gamma_\mu u(p_2)}{(p_1'-p_1)^2}\,, \\[2ex]
M_2 = \dfrac{e^4}{(2\pi)^6}\,\dfrac{m^2}{\sqrt{\varepsilon_1\varepsilon_2\varepsilon_1'\varepsilon_2'}}\int\overline{u}(p_1')\gamma_\nu\,\dfrac{i(p_1+k)-m}{(p_1+k)^2+m^2}\,\gamma_\nu u(p_1)\,\overline{u}(p_2')\gamma_\nu\cdot \\
\qquad\cdot\dfrac{i(p_2-k)-m}{(p_2-k)^2+m^2}\,\gamma_\mu u(p_2)\,\dfrac{\mathrm{d}^4k}{[k^2+\Lambda^2][(p_1'-p_1-k)^2+\Lambda^2]}\,, \\[2ex]
M_3 = \dfrac{e^4}{(2\pi)^6}\,\dfrac{m^2}{\sqrt{\varepsilon_1\varepsilon_2\varepsilon_1'\varepsilon_2'}}\int\overline{u}(p_1')\gamma_\nu\,\dfrac{i(p_1+k)-m}{(p_1+k)^2+m^2}\,\gamma_\mu u(p_1)\cdot \\
\qquad\cdot\overline{u}(p_2')\gamma_\mu\,\dfrac{i(p_2'+k)-m}{(p_2'+k)^2+m^2}\,\gamma_\nu u(p_2)\,\dfrac{\mathrm{d}^4k}{[k^2+\Lambda^2][(p_1'-p_1-k)^2+\Lambda^2]}\,, \\[2ex]
M_4 = -\dfrac{e^2}{4\pi^2 i}\,\dfrac{m^2}{\sqrt{\varepsilon_1\varepsilon_2\varepsilon_1'\varepsilon_2'}}\,\overline{u}(p_1')\gamma_\mu u(p_1)\,\dfrac{\alpha}{2\pi}\,\Pi_F(k^2)\,\overline{u}(p_2')\gamma_\mu u(p_2)\,, \\[2ex]
M_5 = \dfrac{e^2}{4\pi^2 i}\,\dfrac{m^2}{\sqrt{\varepsilon_1\varepsilon_2\varepsilon_1'\varepsilon_2'}}\,\dfrac{\alpha}{2\pi}\,\dfrac{\overline{u}(p_1')\gamma_\mu u(p_1)\,\overline{u}(p_2')\,\Lambda_{\mu F}(p_2', p_2)\,u(p_2)}{(p_1'-p_1)^2}\,, \\[2ex]
M_6 = \dfrac{e^2}{4\pi^2 i}\,\dfrac{m^2}{\sqrt{\varepsilon_1\varepsilon_2\varepsilon_1'\varepsilon_2'}}\,\dfrac{\alpha}{2\pi}\,\dfrac{\overline{u}(p_1')\Lambda_{\mu F}(p_1', p_1)\,u(p_1)\,\overline{u}(p_2')\gamma_\mu u(p_2)}{(p_1'-p_1)^2}\,.
\end{cases}
$$

As the matrix elements M_2, M_3, M_5, M_6 are divergent for small values of momenta of the virtual photons (infrared catastrophe), a small fictitious photon mass Λ is introduced, which will be next eliminated.

The total matrix element is given by the sum of the previous matrix elements plus the exchange matrix elements and can be put in the form

$$M = M^{(1)} + \frac{\alpha}{2\pi}\,M^{(2)}\,.$$

with $M^{(1)}$ the matrix element of order e^2 and $(\alpha/2\pi)M^{(2)}$ the correction. The total transition probability for our process is connected with the absolute square of M:

$$|M|^2 = |M_1|^2 + 2\,\frac{\alpha}{2\pi}\,\mathrm{Re}\,\{M^{(1)}M^{(2)*}\} + \left(\frac{\alpha}{2\pi}\right)^2 |M_2|^2\,.$$

3. – The cross-section.

To determine the differential cross-section for unpolarized electrons the sum over initial and final spin states is required as well as the evaluation of certain integrals. The calculation of the spurs, though quite trivial, has required a long and tedious work. For the integrals we have employed the usual

techniques, whose details are reported in the Appendix. Finally for the cross-section we have, neglecting the e^8 terms,

$$
\begin{aligned}
(2)\quad & \left\{ \mathrm{d}\sigma = \frac{4\alpha^2\beta_1^2\varepsilon_1'\,\mathrm{d}\Omega_1'}{m^2\sqrt{k^2-1}(\beta_1 E - |\bar{P}|\cos\theta_1')} \left\{ \frac{1}{2(\lambda-1)}\frac{1}{2}(k^2+\mu^2+2\lambda-2)\cdot \right.\right. \\
& \cdot\left[\frac{1}{2(\lambda-1)} - \frac{\alpha}{2\pi}\Pi_F\right] + \frac{1}{2(\mu-1)}\frac{1}{2}(k^2-2k)\left[\frac{1}{2(\lambda-1)} - \frac{\alpha}{2\pi}\Pi_F\right] + \\
& + \frac{\alpha}{2\pi}\frac{1}{2(\lambda-1)^2}\cdot[A_1(k^2+\mu^2-2\lambda+2) + mB\cdot 2(\lambda-1)(\lambda-2)] + \\
& + \frac{\alpha}{2\pi}\frac{1}{4(\lambda-1)(\mu-1)}[2A_1(k^2-2k) + mB(\mu-1)(4k-\mu)] - \\
& - \frac{\alpha}{2\pi}\frac{1}{2(\lambda-1)}\left[I_1 k[4k^2+(1-\lambda)(3-\lambda+2k)] + F_1 k(\lambda-3-2k) + \right. \\
& + \frac{1}{2(1+\lambda)}(1+\lambda^2-2\lambda+3k-k\lambda)[H_1(\lambda-1)+\ln 2(\lambda-1)] + \\
& \left. + \tfrac{1}{2}G_1(8k^2+\lambda^2+5-6\lambda+7k-3k\lambda)\right] + \frac{\alpha}{2\pi}\frac{1}{2(\mu-1)}\cdot \\
& \cdot\left[I_1[(\lambda-1)(\mu+2\lambda+1) + 2k(2k-k^2) + \right. \\
& + M_1(1-k) - F_1(1+\mu+2\lambda) + \frac{1}{1+\lambda}[\tfrac{1}{2}H_1(\lambda-1)(1-\lambda-2\mu) + \\
& \left. + (1-\mu)\ln 2(\lambda-1)] + \frac{G_1}{2}[\lambda-7+2(\mu+\lambda)(7-2\mu-2\lambda)]\right] - \\
& - \frac{\alpha}{2\pi}\frac{1}{2(\lambda-1)}\left[I_2\mu[(1-\lambda)(3-\lambda-2\mu)+4\mu^2] + \right. \\
& + F_2\mu(2\mu+\lambda-3) - \frac{1}{2(1+\lambda)}(1+\lambda^2-2\lambda+\mu\lambda-3\mu)\cdot \\
& \left. \cdot[(\lambda-1)H_2 + \ln 2(\lambda-1)] + \tfrac{1}{2}G_2(6\lambda+7\mu-3\mu\lambda-8\mu^2-\lambda^2-5)\right] + \\
& + \frac{\alpha}{2\pi}\frac{1}{2(\mu-1)}\left[I_2[2(1-\mu)(\mu^2-2\mu-1+\mu\lambda) + \lambda(1+\mu+\lambda-\mu\lambda)] - \right. \\
& - F_2[(1+\mu)\lambda + 2(1-\mu)^2] - \frac{1}{2(1+\lambda)}(\mu-1-\mu\lambda-\lambda^2)\cdot \\
& \left. \cdot[H_2(\lambda-1) + \ln 2(\lambda-1)] + \tfrac{1}{2}G_2[\lambda^2-3\lambda+(1-\mu)(9-4\mu-3\lambda)]\right] + \\
& + \text{idem}\begin{pmatrix} k\leftrightarrow k \\ \mu\leftrightarrow\lambda\end{pmatrix} - \frac{\alpha}{2\pi}\ln\frac{\Lambda}{m}\frac{1}{\lambda-1}\cdot\left[\frac{k^2+\mu^2-2\lambda+2}{\lambda-1} + \frac{k^2-2k}{\mu-1}\right]\cdot \\
& \cdot\left[1 + \frac{k}{\sqrt{k^2-1}}\ln\frac{1+k+\sqrt{k^2-1}}{1+k-\sqrt{k^2-1}} + \frac{\lambda}{\sqrt{\lambda^2-1}}\ln\frac{\sqrt{\lambda^2-1}-(\lambda-1)}{\sqrt{\lambda^2-1}+(\lambda-1)} + \right. \\
& \left.\left. + \frac{\mu}{\sqrt{\mu^2-1}}\ln\frac{\sqrt{\mu^2-1}-(\mu-1)}{\sqrt{\mu^2-1}+(\mu-1)}\right] + \text{idem}\begin{pmatrix} k\leftrightarrow k \\ \mu\leftrightarrow\lambda\end{pmatrix}\right\}.
\end{aligned}
$$

The cross-section and, in the following text any quantity of interest, are expressed as functions of the fundamental invariants k, λ, μ defined by

$$(3) \qquad \begin{cases} -m^2 k = p_1 \cdot p_2 = p_1' \cdot p_2' \\ -m^2 \lambda = p_1 \cdot p_1' = p_2 \cdot p_2' \\ -m^2 \mu = p_1 \cdot p_2' = p_1' \cdot p_2 \end{cases}$$

and the other symbols have the usual meaning. Moreover

$$E = \varepsilon_1 + \varepsilon_2 , \qquad \boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 .$$

For energy-momentum conservation k, λ, μ are related by the relation

$$\mu + \lambda = 1 + k .$$

Π_F derives from the photon self-energy part that has been previously referred.

A_1, mB derive from the vertex parts and are obtained from the integrals of $\Lambda_{\mu F}$.

$$A_1 = -2 + \frac{1}{2\sqrt{\lambda^2-1}} \ln \frac{\sqrt{\lambda^2-1}+(\lambda-1)}{\sqrt{\lambda^2-1}-(\lambda-1)} [1 + 3\lambda - \lambda \ln 2(\lambda+1)] +$$

$$+ \bar{\mathscr{L}}\left(\frac{\sqrt{\lambda^2-1}+(\lambda-1)}{2\sqrt{\lambda^2-1}}\right) - \bar{\mathscr{L}}\left(\frac{\sqrt{\lambda^2-1}-(\lambda-1)}{2\sqrt{\lambda^2-1}}\right) =$$

$$= A + 2 \ln \frac{\Lambda}{m} \left[1 - \frac{\lambda}{\sqrt{\lambda^2-1}} \ln \frac{\sqrt{\lambda^2-1}+(\lambda-1)}{\sqrt{\lambda^2-1}-(\lambda-1)}\right],$$

$$mB = \frac{1}{2\sqrt{\lambda^2-1}} \ln \frac{\sqrt{\lambda^2-1}+(\lambda-1)}{\sqrt{\lambda^2-1}-(\lambda-1)},$$

where

$$\bar{\mathscr{L}}(x) = \int_0^x \frac{\ln|1-u|}{u} \, du .$$

The quantities I_1, F_1, G_1, H_1 are related to the integrals appearing in M_2 and M_3 and their expression is:

$$I_1 = -\frac{1}{2(\lambda-1)\sqrt{k^2-1}} \ln 2(\lambda-1) \ln \frac{1+k+\sqrt{k^2-1}}{1+k-\sqrt{k^2-1}},$$

$$F_1 = L_1 = -\frac{1}{\sqrt{k^2-1}}\left[\frac{\pi^2}{4} + \frac{1}{4}\ln 2(k-1)\ln\frac{1+k+\sqrt{k^2-1}}{1+k-\sqrt{k^2-1}} + \right.$$

$$\left. - \frac{1}{2}\bar{\mathscr{L}}\left(\frac{1+k+\sqrt{k^2-1}}{2\sqrt{k^2-1}}\right) - \frac{1}{2}\bar{\mathscr{L}}\left(\frac{\sqrt{k^2-1}-(k+1)}{2\sqrt{k^2-1}}\right)\right],$$

$$G_1 = H_1 = -\frac{1}{2\sqrt{\lambda^2-1}}\left[\ln 2(\lambda-1)\ln\frac{\sqrt{\lambda^2-1}-(\lambda-1)}{\sqrt{\lambda^2-1}+(\lambda-1)} + \right.$$

$$+ \bar{\mathscr{L}}(\lambda-\sqrt{\lambda^2-1}) - \bar{\mathscr{L}}(\lambda+\sqrt{\lambda^2-1}) + 2\bar{\mathscr{L}}\left(\frac{\sqrt{\lambda^2-1}+(\lambda-1)}{2(\lambda-1)}\right) -$$

$$\left. - 2\bar{\mathscr{L}}\left(\frac{\sqrt{\lambda^2-1}-(\lambda-1)}{2(\lambda-1)}\right)\right],$$

$$M_1 = \frac{1}{2\sqrt{\mu^2-1}}\ln\frac{\sqrt{\mu^2-1}+(\mu-1)}{\sqrt{\mu^2-1}-(\mu-1)}.$$

Therefore, finally

$$\mathrm{d}\sigma = \mathrm{d}\sigma^{(1)} + \frac{\alpha}{2\pi}\,\mathrm{d}\sigma^{(2)} = \mathrm{d}\sigma^{(1)}(1+\delta),$$

where

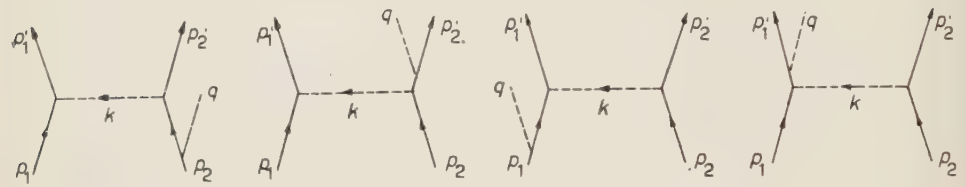
$$\delta = +\frac{\alpha}{2\pi}\frac{\mathrm{d}\sigma^{(2)}}{\mathrm{d}\sigma^{(1)}}.$$

4. – The infrared catastrophe and the bremsstrahlung cross-section.

The integral A relative to the vertex-part and the ones relative to M_2, M_3 are logarithmically divergent for small k. The origin of such a divergence is very well known. It can be eliminated considering that the e^--e^- scattering always occurs with emission of real photons, because the electrons are accelerated during the interaction. From an experimental view-point these photons cannot be detected unless their energies are larger than the energy resolution $\Delta\varepsilon$. In any case, even if this process is not experimentally accessible, it must not be omitted. Thus we have to add to the corrected cross-section (which describes the emission of virtual photons only) that for emission of real photons with energy $< \Delta\varepsilon$. The recombination of the two processes completely eliminates infrared divergences, to all orders in e.

To this end let us consider the bremsstrahlung process associated with Møller scattering.

The emission of a soft real photon of momentum $q(\boldsymbol{q}, \sqrt{\overline{q}^2+\Lambda^2})$ and polarization « e » ($\omega \ll m$), is described, to the order e^3, by the diagrams



and by as many others for the exchange. With the condition $\omega \ll m$, the total matrix element can be put in the form:

$$M_1 = -\frac{e}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2\omega}}\,\overline{u}(p_2')\,\overline{u}(p_1')\,M\,u(p_2)\,u(p_1)\cdot$$

$$\cdot\left[\frac{p_2'\cdot e}{p_2'\cdot q}+\frac{p_1'\cdot e}{p_1'\cdot q}-\frac{p_2\cdot e}{p_2\cdot q}-\frac{p_1\cdot e}{p_1\cdot q}\right], \qquad \omega=\sqrt{\overline{q}^2+\Lambda^2},$$

where M is the e^2 matrix element for Møller scattering. Therefore the bremsstrahlung cross-section can be obtained by multiplying the Møller cross-section by the factor:

$$B=\frac{e^2}{(2\pi)^3}\int\frac{\mathrm{d}^3q}{2\omega}\sum_{\mathrm{pol}}\left(\frac{p_2'\cdot e}{p_2'\cdot q}+\frac{p_1'\cdot e}{p_1'\cdot q}-\frac{p_2\cdot e}{p_2\cdot q}-\frac{p_1\cdot e}{p_1\cdot q}\right)^2.$$

This expression, when summed over polarizations, is to be integrated with respect to the photon momentum over all angles and from $|\overline{q}|=0$ to $|\overline{q}|\Delta\varepsilon$ with $\Delta\varepsilon \ll m$. This calculation can be conveniently performed in C.M. where

$$k = 2\gamma^2 - 1$$

$$\lambda = \gamma^2(1+\beta^2\cos\theta)$$

$$\mu = \gamma^2(1-\beta^2\cos\theta)$$

and θ is the angle between the incident electron and the recoil electron. Re-expressing then all the quantities in terms of the invariants μ, k, λ we obtain

for the cross-section the following form valid in C.M. reference frame:

$$(4)\qquad d\sigma_{\text{Bremss}} = \frac{4\alpha^2\beta_1^2 d\Omega_1'}{m^2\sqrt{k^2-1}}\,\frac{\varepsilon_1'}{(\beta_1' E - |\bar{P}|\cos\theta_1')}\,\frac{\alpha}{2\pi}\cdot$$

$$\cdot\left\{\frac{1}{(\lambda-1)^2}\,\frac{1}{2}\,(k^2+\mu^2-2\lambda+2)+\frac{1}{(\lambda-1)(\mu-1)}\,\frac{1}{2}\,(k^2-2k)+\text{idem}\begin{pmatrix}k\leftrightarrow k\\ \mu\leftrightarrow\lambda\end{pmatrix}\right\}\cdot$$

$$\cdot\left\{\sqrt{\frac{k+1}{k-1}}\ln\frac{1+k+\sqrt{k^2-1}}{1+k-\sqrt{k^2-1}}-\lambda G(k,\lambda)-\mu G(k,\mu)+kG(k,k)-\right.$$

$$-2\ln\frac{2\Delta\varepsilon}{\Lambda}\left[1+\frac{k}{\sqrt{k^2-1}}\ln\frac{1+k+\sqrt{k^2-1}}{1+k-\sqrt{k^2-1}}+\frac{\mu}{\sqrt{\mu^2-1}}\cdot\right.$$

$$\left.\left.\cdot\ln\frac{\sqrt{\mu^2-1}-(\mu-1)}{\sqrt{\mu^2-1}+(\mu-1)}+\frac{\lambda}{\sqrt{\lambda^2-1}}\ln\frac{\sqrt{\lambda^2-1}-(\lambda-1)}{\sqrt{\lambda^2-1}+(\lambda-1)}\right]\right\},$$

and $G(k,\lambda)$ is so defined

$$G(k,\lambda)=\frac{2}{\sqrt{k-1}\sqrt{\lambda-1}}\int_{1/(k-\lambda)(k-1)}^{1}\frac{dz}{\sqrt{\left(1-z^2\frac{k-1}{k+1}\right)\left(z^2-\frac{k-1}{\lambda-1}\right)}}\ln\frac{1+z\sqrt{\frac{k-1}{k+1}}}{1-z\sqrt{\frac{k-1}{k+1}}},$$

and the multiplication constant is to be expressed in C.M. system ($\bar{P}=0$).

Such an integral can be analytically evaluated only in a few special cases and in general it must be numerically evaluated.

For instance for $k=\lambda$, we have

$$G(k,k)=\frac{1}{k}\left(\frac{\pi^2}{3}+\frac{1}{2}\ln^2 2k\right)\cdot\qquad k\gg 1.$$

Adding now the (4) and the (2) we see at once that in the latter $\ln \Lambda/m$ is substituted with $\ln 2\Delta\varepsilon/m$. Besides we have to add the bremsstrahlung-terms not depending from Λ. By doing so only quantities of full physical meaning occur in the cross-section.

The previous results can be applied to positron-electron scattering with the « substitution law » (see [12])

$$p_1\to p\,,\qquad p_1'\to p'\,,\qquad p_2\to -q'\,,\qquad p_2'\to -q\,.$$

where p, p', q, q' are the initial and final momenta of the negative and positive electrons respectively. In particular the k, λ, μ invariants are now defined as

$$\begin{aligned}-m^2k &= -p\cdot q' = -p'\cdot q\\ -m^2\lambda &= p\cdot p' = q\cdot q'\\ -m^2\mu &= -p\cdot q = -p'\cdot q'\,.\end{aligned}$$

A special attention, however, must be dedicated in this case to the behaviour of the $\mathscr{L}(x)$ trascendental functions. Numerical computations and results for Bhabha scattering will be published in a next paper.

5. – Numerical results.

For Møller scattering the three quantities k, λ, μ take in the laboratory system the following simple meaning

$$k = \frac{\varepsilon_2}{m} = \gamma\,, \qquad \lambda = \frac{\varepsilon_1'}{m}\,, \qquad \mu = \frac{\varepsilon_2'}{m}\,,$$

where ε_2 is the energy of the incident electron, ε_1' of the recoil electron and ε_2' of the scattered electron. By introduction of θ angle (between the incident and recoil electron) we have

$$\lambda = \frac{(\gamma+1)+(\gamma-1)\cos^2\theta}{(\gamma+1)-(\gamma-1)\cos^2\theta}\,,$$

$$\mu = \frac{(\gamma+1)[(\gamma+1)-(\gamma-1)\cos^2\theta]-[(\gamma+1)+(\gamma-1)\cos^2\theta]}{(\gamma+1)-(\gamma-1)\cos^2\theta}\,.$$

To give some idea of the order of magnitude of the « radiative corrections » we have chosen a situation of maximum non-local effects. This occurs when both particles in L.S. leave the reaction with equal energies. This is expressed by the condition

$$\mu = \lambda$$

and it corresponds to an angle $\theta = \pi/2$ in C.M. and to angle in L.S.

$$\cos^2\theta_L = \frac{\gamma+1}{\gamma+3}\,.$$

Assuming for k the values 200, 2 000, 20 000 corresponding to energies of 100, 1 000, 10 000 MeV in the L.S. ($\simeq 5$, 16, 50 MeV in the C.M.S.) the result for $\delta = (\alpha/2\pi)(\mathrm{d}\sigma^{(2)}/\mathrm{d}\sigma^{(1)})$ is $\simeq -0.05$, -0.11, -0.22.

We point out that the situation which we consider should yield a maximum value for the « radiative corrections ». In fact the terms corresponding to direct scattering and to exchange scattering are equal in this case. In comparison with the data of BROWN and FEYNMAN [10] the maximum value of the corrections for the Compton effect is about -9% at 1000 MeV, *i.e.* of the same order as our corresponding result.

The evaluation has been made using formula (2) and leaving out the terms proportional to $\ln \Lambda/m$. To neglect the further terms deriving from adding the bremsstrahlung cross-section (4) is utterly justified. It may be observed, in fact, that formula (4) which we give will thus be correct only if actual energies fall within the restriction $\Delta\varepsilon \ll m$ (note that (4) is of course valid in the C.M. system because $\Delta\varepsilon \ll m$ is not a covariant condition). Otherwise it is necessary to use the rigorous formula for photon emission in the electron-electron scattering, integrated for photon momentum from 0 to $\Delta\varepsilon$.

* * *

In conclusion the authors wish to express their appreciation to Prof. P. BUDINI for many discussions and to Dr. GRATZER for her help in calculations.

APPENDIX

Evaluation of the integrals.

a) *Vertex-part.* – In this case the integrals to be calculated are double or triple integrals in the auxiliary variables x, y and z. Their computation is not difficult and provides the result referred. We have besides

$$C = 0 .$$

In the expression for A, B there occur the so called Spence functions defined as

$$\mathscr{L}(x) = \int_0^x \frac{\ln (1-t)}{t}\, dt ,$$

and frequently used in the following text. It seems in fact that the knowledge of such functions suffices for the evaluation of radiative corrections in any case (*). Moreover, since in the cross-section there occur only real quantities, where $x > 1$ we have to take $\ln (|1-t|)$.

b) Following Feynman-notations it is convenient to put

$$(1) = k^2 + 2p_1 \cdot k , \qquad (2) = k^2 - 2p_2 \cdot k , \qquad (2') = k^2 + 2p_2' \cdot k ,$$

$$(3) = k^2 + p_3^2 + \Lambda^2 - 2p_3 \cdot k , \qquad (0) = k^2 + \Lambda^2 ,$$

(*) For numerical computation of this integrals see K. MITHCHELL: *Phil. Mag.*, **40**, 351 (1949).

and the integrals from the matrix elements M_2, M_3 can be written

$$I_1(1, \varrho, \varrho\sigma) = \int \frac{(1, k_\varrho, k_\varrho k_\sigma)\, \mathrm{d}^4k}{(1)(2)(3)(0)},$$

$$I_2(1, \varrho, \varrho\sigma) = \int \frac{(1, k_\varrho, k_\varrho k_\sigma)\, \mathrm{d}^4k}{(1)(2')(3)(0)}.$$

In the following text there occur also these integrals with three denominators:

$$F_{1(1\sigma)} = \int \frac{(1, k_\sigma)\, \mathrm{d}^4k}{(1)(2)(3)}, \qquad G_{1(1,\sigma)} = \int \frac{(1, k_\sigma)\, \mathrm{d}^4k}{(2)(3)(0)},$$

$$H_{1(1.\sigma)} = \int \frac{(1, k_\sigma)\, \mathrm{d}^4k}{(1)(3)(0)}, \qquad L_{1(1.\sigma)} = \int \frac{(1, k_\sigma)\, \mathrm{d}^4k}{(1)(2)(0)},$$

and analogously F_2, G_2 with the exchange $(2) \leftrightarrow (2')$, $-p_2 \leftrightarrow p_2'$. This corresponds in terms of k, λ, μ to

$$\lambda \leftrightarrow \lambda, \qquad k \leftrightarrow -\mu,$$

and so the integrals with denominator $(2')$ can be obtained from that with denominator (2), with the above exchange.

With the usual notations we obtain for the integrals:

$$I_1 = \frac{\pi^2 i}{2m^4(\lambda-1)\sqrt{k^2-1}} \ln \frac{1+k+\sqrt{k^2-1}}{1+k-\sqrt{k^2-1}} \left[2 \ln \frac{\Lambda}{m} - \ln 2(\lambda-1) \right],$$

$$G_1 = H_1 = \frac{\pi^2 i}{2m^2\sqrt{\lambda^2-1}} \Big[\ln 2(\lambda-1) \ln \frac{1-\lambda-\sqrt{\lambda^2-1}}{1-\lambda+\sqrt{\lambda^2-1}} + \mathscr{L}(\lambda+\sqrt{\lambda^2-1}) -$$
$$- \mathscr{L}(\lambda-\sqrt{\lambda^2-1}) - 2\mathscr{L}\left(\frac{\sqrt{\lambda^2-1}+\lambda-1}{2(\lambda-1)}\right) + 2\mathscr{L}\left(\frac{\sqrt{\lambda^2-1}-(\lambda-1)}{-2(\lambda-1)}\right) \Big],$$

$$F_1 = L_1 = \frac{\pi^2 i}{m^2} \Big\{ \ln \frac{\Lambda}{m} \ln \frac{1+k-\sqrt{k^2-1}}{1+k+\sqrt{k^2-1}} + \frac{1}{4} \cdot$$
$$\cdot \Big[\pi^2 + i\pi \ln 2(1+k) - \ln 2(k-1) \ln \frac{1+k-\sqrt{k^2-1}}{1+k-\sqrt{k^2-1}} -$$
$$- i\pi \ln \frac{k-1}{k+1} + 2\mathscr{L}\left(\frac{1+k+\sqrt{k^2-1}}{2\sqrt{k^2-1}}\right) - 2\mathscr{L}\left(\frac{\sqrt{k^2-1}-(1+k)}{2\sqrt{k^2-1}}\right) \Big] \Big\} \frac{1}{\sqrt{k^2-1}}.$$

(Note that for the cross-section only the imaginary part of the integrals is needed).

$$I_{1\sigma} = a_1 p_{1\sigma} + b_1 p_{2\sigma} + c_1 p_{3\sigma}\,, \qquad p_3 = p_1' - p_1,$$

$$a_1 = -\frac{(G_1 - F_1)}{2m^2(k-\lambda)} + \frac{1}{2}\frac{\lambda-1}{k-\lambda} I_1\,,$$

$$b_1 = -\frac{(G_1 - F_1)}{2m^2(k-\lambda)} - \frac{1}{2}\frac{\lambda-1}{k-\lambda} I_1 = -a_1\,,$$

$$c_1 = \frac{G_1 - F_1}{2m^2(k-\lambda)} + \frac{1}{2}\frac{k}{k-\lambda} - \frac{1}{2} I_1\,,$$

$$I_{1\varrho\sigma} = \alpha_{1\varrho} p_{1\sigma} + \beta_{1\varrho} p_{2\sigma} + \gamma_{1\varrho} p_{3\sigma} + \varepsilon_1 \delta_{\varrho\sigma}\,,$$

$$\alpha_{1\varrho} = \frac{1}{4m^2(1+k)(k-\lambda)}\Big\{-kF_{1\varrho} + G_{1\varrho}(1+\lambda) + H_{1\varrho}(\mu+k) + (1+k)\cdot$$
$$\cdot(2m^2(\lambda-1)I_{1\varrho} - L_{1\varrho}) - 2\varepsilon_1[p_{1\varrho}(1+\lambda) - p_{2\varrho}(\mu+k) + p_{3\varrho}(1+k)]\Big\},$$

$$\beta_{1\varrho} = \frac{1}{4m^2(1+k)(k-\lambda)}\Big\{kF_{1\varrho} - G_{1\varrho}(\mu+k) - H_{1\varrho}(1+\lambda) - (1+k)\cdot$$
$$\cdot(2m^2(\lambda-1)I_{1\varrho} - L_{1\varrho}) + 2\varepsilon_1[(\mu+k)p_{1\varrho} - p_{2\varrho}(1+\lambda) + p_{3\varrho}(1+k)]\Big\},$$

$$\gamma_{1\varrho} = \frac{1}{4m^2(1-\lambda)(k-\lambda)}\Big\{F_{1\varrho}(\lambda-\mu) + (1-\lambda)(G_{1\varrho} - H_{1\varrho}) +$$
$$+ (1-k)(2m^2(\lambda-1)I_{1\varrho} - L_{1\varrho}) - 2\varepsilon_1[(1-\lambda)(p_{1\varrho} - p_{2\varrho}) + p_{3\varrho}(1-k)]\Big\},$$

$$\varepsilon_1 = \frac{F_1}{2}\frac{k-1}{k-\lambda} - \frac{G_1}{2}\frac{\lambda-1}{k-\lambda} - \frac{m^2}{2} I_1 \frac{(k-1)(\lambda-1)}{k-\lambda}\,.$$

In the cross-section (2) we have used the same denomination for quantities which, though deriving from these integrals, are a little different.

$$F_{1\sigma} = p_{3\sigma} F_1 + \frac{\pi^2 i}{m^2}\Big\{(p_2 - p_3)_\sigma \frac{1}{\sqrt{k^2-1}} \ln\frac{1+k-\sqrt{k^2-1}}{1+k+\sqrt{k^2-1}} -$$
$$-\frac{(p_1+p_2)_\sigma}{2}\Big[\frac{i\pi}{1+k} + \frac{1}{\sqrt{k^2-1}} \ln\frac{1+k-\sqrt{k^2-1}}{1+k+\sqrt{k^2-1}}\Big]\Big\},$$

$$G_{1\sigma} = \frac{p_{2\sigma}}{1+\lambda}\Big[G_1(\lambda-1) - \frac{\pi^2 i}{m^2}\ln 2(\lambda-1)\Big] + \frac{p_{3\sigma}}{1+\lambda}\Big[G_1 - \frac{\pi^2 i}{2m^2}\ln 2(\lambda-1)\Big],$$

$$H_{1\sigma} = G_{1\sigma}(p_1 \leftrightarrow -p_2)\,,$$

$$L_{1\sigma} = (p_1 - p_2)_\sigma \frac{\pi^2 i}{2m^2}\Big[\frac{1}{\sqrt{k^2-1}} \ln\frac{1+k-\sqrt{k^2-1}}{1+k+\sqrt{k^2-1}} - i\pi\Big].$$

c) Let us mention briefly the evaluation of $I_{1(1)}$ which, though substantially analogous to that given by Feynman, shows another difficulty. This occurs for a double infrared divergence, depending on the presence of two photon-propagators in the denominator.

By usual means the denominators are collected by Feynman's identity and once we have integrated in d^4k, we arrive at

$$I_{1(1)} = \int \frac{\mathrm{d}^4k}{(1)(2)(3)(0)} = \pi^2 i \int_0^1\int\int \frac{z^2(1-x)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{[a^2zx + \Lambda^2(1-z) - z^2p_x^2]^2},$$

$$p_x = p_3x + p_y(1-x)\,,$$

$$p_y = p_2y - p_1(1-y)\,. \qquad a^2 = p_3^2 + \Lambda^2,$$

The double infrared divergence makes the integral in x divergent not only for $x = 0$ but also for $x = 1$ (*if* $\Lambda = 0$). Taking this into account the x integration can be executed in the following way

$$\int_0^1 \mathrm{d}x = \int_0^\eta + \int_\eta^{1-\delta} + \int_{1-\delta}^1 \qquad \delta, \eta \ll 1\,.$$

In the first integral we keep $\Lambda \neq 0$ everywhere and neglect x with respect to 1. In the second integral we put directly $\Lambda = 0$. In the third integral we may perform the following substitution $x \to 1 - x$, put $\Lambda = 0$, except in a^2, still neglecting x with respect to 1. By integrating it is easily seen that the dependence on δ, η disappears furnishing the given result.

For $I_{1(3)}$ and $I_{1(3\sigma)}$ the method is that given by Feynman in [10].

RIASSUNTO

Sono state calcolate le correzioni di ordine e^6 alla sezione d'urto differenziale per scattering di Møller. Le divergenze per grandi valori dell'impulso k dei fotoni virtuali sono state eliminate secondo i metodi consueti della rinormalizzazione. La divergenza infrarossa è stata tolta aggiungendo la sezione d'urto per il processo di bremsstrahlung associato allo scattering di Møller. Valutazioni numeriche forniscono valori attendibili per tali correzioni.

LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

Further Experimental Evidence Concerning the Fermi-Teller «Z-Law» (*).

J. F. Lathrop, R. A. Lundy, R. A. Swanson,
V. L. Telegdi and D. D. Yovanovitch

The Enrico Fermi Institute for Nuclear Studies
The University of Chicago - Chicago, Ill.

(ricevuto il 3 Agosto 1959)

An investigation performed in this laboratory and published in this Journal (1) showed that in a variety of insulators the so-called Fermi-Teller «Z-law» does not hold. This «law» would predict that negative particles, such as the muons used in our experiments, are captured by the various atomic species Z_1, Z_2, ... of a compound in the ratios $n_1Z_1 : n_2Z_2 : \ldots$, where n_1, n_2, etc., are the stoichiometric proportions of the atoms and Z_i their atomic numbers. Contrary to this prediction, the capture ratios in the insulators investigated were found to be much better approximated by $n_1 : n_2 : \ldots$, *i.e.* by the stoichiometric proportions unweighted with the atomic numbers.

As a possible explanation of these observations, it was suggested in ref. (1) that Fermi and Teller (2) based their predictions — which were supposedly applicable to insulating compounds — on a law for the energy loss (near zero kinetic energy) derived by them for *metals*. While in the latter the energy loss in the lattice plays no major role in the energy region of interest, this mechanism may become very significant for insulators where there exists a Brillouin gap. The energy loss to the lattice, although difficult to calculate, has no obvious dependence on the Z_i's, and thus the weighting predicted strictly speaking only for *metallic* compounds should not arise in insulators.

Fig. 1. shows the time distribution of decay electrons observed from negative muons stopped in the binary compound AgZn. According to the stoichiometric proportions, equal numbers of muons should be captured in Ag and Zn, while the «Z-law» would predict a ratio Ag/Zn=1.57. The intercepts of the Ag and Zn decay curves at $t=0$ are proportional to the actual atomic capture ratio and to the ratio of the decay constants $\lambda_d(Z_i)$ for the K-orbits about the Ag and Zn nuclei. Recent investi-

(*) Research supported by a joint program of the Office of Naval Research and the U. S. Atomic Energy Commission.

(1) J. C. Sens, R. A. Swanson, V. L. Telegdi and D. D. Yovanovitch: *Nuovo Cimento*, **7**, 536 (1958).

(2) E. Fermi and E. Teller: *Phys. Rev.*, **72**, 399 (1947).

gations [3] have shown that these two decay constants, while different from the decay constant for $Z=0$, should be essentially equal for Ag and Zn. The observed intercept ratio Ag/Zn is 2.2 ± 0.7. Thus for this alloy the Fermi-Teller background consisted mostly of a carbon component originating from muons which inevitably stopped in the counter wrappings.

While the AgZn measurement just described appears to bear out our con-

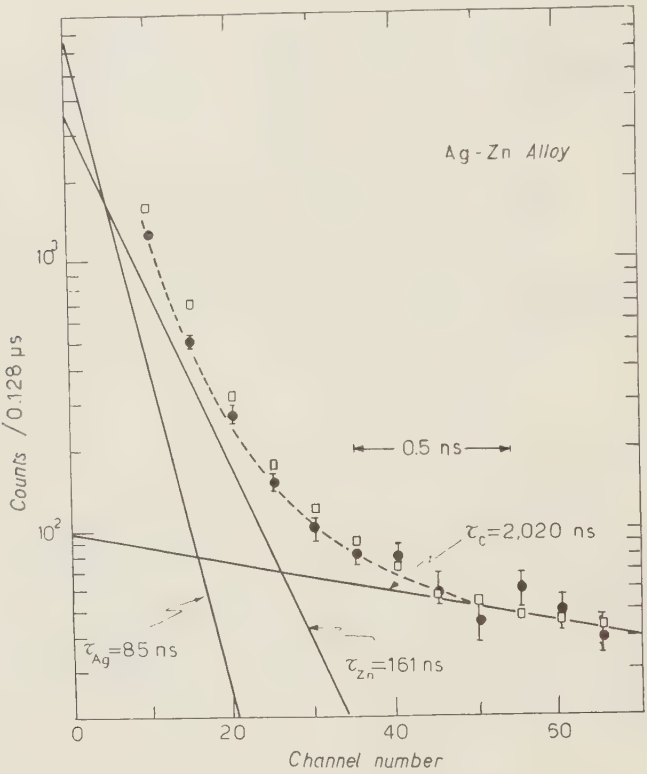


Fig. 1. – Time distribution of electrons from μ^- decaying in AgZn. ● Experimental points. □ are the predicted points assuming equal capture in Ag and Zn. Errors are statistical standard deviations. Indicated background has the slope of decays in carbon.

prediction is supported by experiment; the inconsistency of the data with the stoichiometric ratio is illustrated by the points indicated as □ in Fig. 1, which represent the experimental points which should be observed for this capture ratio. In this measurement which, contrary to ref. [1], involved elements of high Z, the jecture about the validity of the Fermi-Teller « Z-law » for metallic compounds, we have also recently obtained data on an insulating compound which are in flat contradiction with the behavior observed for many insulators in ref. [1]. In view of the interest in determining the rate of the process $^6\mathrm{Li}+\mu=^6\mathrm{He}+\nu$, we have investigated the capture in a crystal of ^{6}LiI. Fig. 2 shows the time distribution of decay electrons from μ^- stopped in such a crystal sealed in

[3] R. A. Lundy, J. C. Sens, R. A. Swanson, V. L. Telegdi and D. D. Yovanovitch: *Phys. Rev. Lett.*, **1**, 205 (1958).

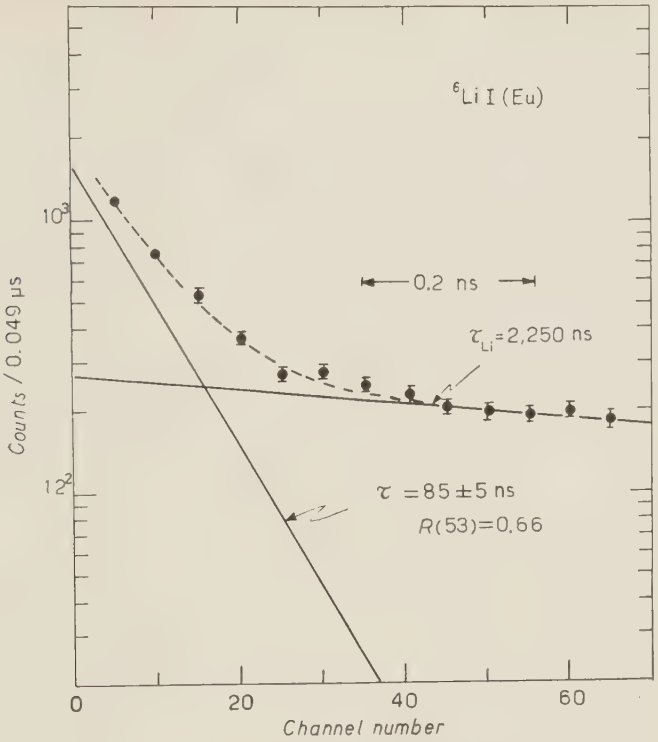


Fig. 2. – Time distribution of electrons from μ^- decaying in ^{6}LiI. Experimental points. Errors are statistical standard deviations.

a thin aluminum container with a glass window on the beam entrance side. Fig. 3 is a range curve for the pion-muon beam used, and gives an indication of the magnitude of the fraction of muons expected to stop in parts of the container. From the $t=0$ intercepts alone, an atomic capture ratio I/Li$=$ $=6.9\pm0.5$ is estimated. This is however only a lower limit, inasmuch as we know [4] that $\lambda_d(\mathrm{I})/\lambda_d(\mathrm{Li})\simeq 0.66$. Taking this factor into account, one has I/Li$=$ $=10.5\pm1.3$. Further corrections must be applied for the decays from stops in the container walls, and lead to a final ratio I/Li$=15.8\pm2.0$. This ratio is obviously not the stoichiometric one, but

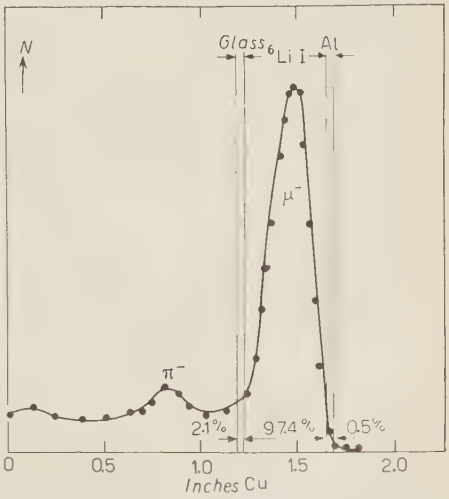


Fig. 3. – Range distribution of the 150 MeV/c pion-muon beam used. Vertical lines indicate the surfaces of the ^{6}LiI crystal and of the container walls. The indicated percentages are estimates of the relative number of muons stopping in the various materials indicated.

[4] D. D. YOVANOVITCH: to be published in the *Phys. Rev.*

is rather well approximated by the Fermi-Teller prediction I/Li = 17.6. It is not clear why the « Z-law » should apply to LiI and not to the other insulators investigated by us previously [1], although it is worth pointing out that none of the compounds tested earlier had constituents of as widely different atomic numbers as LiI nor contained as light an atom as Li. The high I/Li ratio observed could also be a consequence of transfer processes such as have been suggested [5] to explain the anomalous X-ray yields observed by the Stearnses [6].

The experimental I/Li ratio found here makes ^{6}LiI a rather unsuitable medium for the μ-capture experiment referred to above. This ratio is at variance with the result of a similar measurement recently performed elsewhere [7], by a method less amenable to simple analysis than the data shown in Fig. 2.

[5] T. B. Day and P. Morrison: *Phys. Rev.*, **107**, 912 (1957).

[6] M. B. Stearns and M. Stearns: *Phys. Rev.*, **105**, 1573 (1957).

[7] G. Backenstoss, B. Bloch, B. Chidley, R. Reiter, T. Romanowski, R. Siegel and R. Sutton: *Bull. Am. Phys. Soc.*, II, **4**, 273 (1959).

On the Distribution of Energy in the Schwarzschild Field.

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(ricevuto il 16 Dicembre 1959)

Recently Arnowitt, Deser and Misner [1] have proposed a new canonical formalism for general relativity which enables them to assign a Hamiltonian density to the gravitational field when certain coordinate conditions are imposed [2]. It is the hope of such an approach to lay the groundwork for quantizing the general theory using the well-known techniques of the Hamiltonian formalism, and treating the gravitational field in a manner similar to the electromagnetic field. We wish to anticipate in this note some of the difficulties this approach will encounter by giving expressions for the metric of the Schwarzschild field in their coordinate system. As we shall see, both the metric and the energy density become complex and highly singular near the origin, thereby invalidating certain conclusions obtained from the theory. As a supplement to the above considerations, we shall also describe briefly an alternative approach to obtaining an energy-momentum tensor for the gravitational field. The method will be based primarily on the requirement that this quantity transform as a true tensor under *all* coordinate transformations.

To obtain the metric of the Schwarzschild field in the A.D.M. formalism we satisfy one of the coordinate restrictions by introducing a time-orthogonal static system of coordinates. In this case the coordinate restrictions reduce to

$$g_{ij,j} = 0\,, \qquad (i, j = 1, 2, 3), \tag{1}$$

together with the boundary conditions that the coordinate system becomes rectangular at infinity [3]. If we write the line-element in the quasi-rectangular form

$$\mathrm{d}s^2 = g_{00}\,\mathrm{d}t^2 + \left[\frac{R^2}{r^2}\,\delta_{ij} + \left(\dot{Q} - \frac{R^2}{r^2}\right)\frac{x^i x^j}{r^2}\right]\mathrm{d}x^i\,\mathrm{d}x^j, \tag{2}$$

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(1) R. Arnowitt, S. Deser and C. W. Misner: *Nuovo Cimento*, **15**, 487 (1960), see also, R. Arnowitt and S. Deser: *Phys. Rev.*, **113**, 745 (1959).

(2) These coordinate conditions are: $g_{ij,j} = 0$, $(i, j = 1, 2, 3)$ $\Pi^{ij}{}_{,jj} - \Pi^{ij}{}_{,ij} = 0$, where $\Pi^{ij} \equiv \sqrt{-g}\cdot[\Gamma^0_{lm} - g_{lm}\,\Gamma_{pq}\,\gamma^{pq}]\,\gamma^{li}\,\gamma^{mj}$; γ^{ij} is the matrix inverse to g_{ij}; a comma denotes ordinary differentiation.

(3) Without these boundary conditions, the coordinate conditions are satisfied, for example, by polar coordinates in flat space, for which the formalism yields meaningless results. One is also restricted, *a fortiori*, to cosmologies in which space is flat in the large.

with $r = (x^i x^i)^{\frac{1}{2}}$, $R = R(r)$, $Q = Q(r)$ the restrictions (1) reduce to

$$P_{,v} = 2R^2 , \tag{3}$$

where $v \equiv \ln r$, $P \equiv r^2 Q$. The field equations, together with the boundary conditions that the metric be asymptotically flat at infinity yield the following additional relation [4]

$$P = R^2_{,v}(1 - \alpha/R)^{-1} , \qquad (\alpha \equiv 2GM/rc^2) , \tag{4}$$

together with the expression $-g_{00} = (1 - \alpha/R)$ which, however, plays no role in the following discussion. Eqs. (3) and (4) may be integrated to give $\ln (r/\alpha)$ in terms of R as follows [5]

$$\ln (r/\alpha) = \int [3I(R)]^{-\frac{1}{3}}(1 - \alpha/R)^{-\frac{1}{2}} \, dR , \tag{5}$$

where $I(R)$ is given by the integral

$$\begin{aligned} I(R) &= \int R^2(1 - \alpha/R)^{-\frac{1}{2}} \, dR = \\ &= \left(1 - \frac{\alpha}{R}\right)^{\frac{1}{2}} \left\{ -\frac{R^3}{3} + \frac{5\alpha R^2}{12} - \frac{15}{24}\alpha^2 R \right\} - \frac{15}{24}\alpha^3 \operatorname{tgh}^{-1}\left(1 - \frac{\alpha}{R}\right)^{\frac{1}{2}} . \end{aligned} \tag{6}$$

For computational purposes (5) is too complicated, and it is more convenient to resort to approximation techniques to obtain R and Q in the regions of interest, $r \gg \alpha$, $r \approx \alpha$, $r \ll \alpha$. The solutions for the various cases are:

$$r \gg \alpha \begin{cases} \dfrac{R^2}{r^2} = 1 + \dfrac{\alpha}{2r} + \dfrac{\alpha^3}{r^3}\left\{\dfrac{27}{32}\ln (r/\alpha) - \dfrac{1}{32}\right\} + \dots , \\ Q = 1 + \dfrac{\alpha}{r} + \dfrac{\alpha^2}{r^2} - \dfrac{\alpha^3}{r^3}\left\{\dfrac{27}{16}\ln (r/\alpha) + \dfrac{26}{16}\right\} + \dots , \end{cases} \tag{7}$$

$$r \approx \alpha \begin{cases} R^2 = \alpha^2\left[1 - \dfrac{9}{2}(\ln r/\alpha)^3\right]^{-2} , \\ Q = 2\dfrac{\alpha^2}{r^2}\ln (r/\alpha) , \end{cases} \tag{8}$$

$$r \ll \alpha \begin{cases} (R/r)^2 = (\alpha/r)^{2-2\omega} , \\ Q = \omega^2(\alpha/r)^{2-2\omega} , \end{cases} \tag{9}$$

where in (9), ω is either one of the complex cube roots of unity.

[4] It is convenient to integrate the equations using the form given in L. Landau and E. Lifschitz: *The Classical Theory of Fields*, p. 311 (Cambridge, Mass., 1951). Note that without the coordinate conditions (1), upon setting $R = r$, we obtain from (4) the usual expressions for the Schwarzschild metric in quasi-rectangular coordinates.

[5] A. D. M. have also previously obtained a solution in quadratures (communication by S. Deser).

Because of this complex behavior of the metric near the origin, the energy density will also be complex in this region. According to the formalism, this quantity is given by $-g^T_{ii}$, where $g^T \equiv g_{ii} - (1/\nabla^2)g_{ij,ij}$, and $(1/\nabla^2)$ is the reciprocal of the flat-space Laplacian operator (note, by the coordinate conditions (1), $\nabla^2 g^T = \nabla^2 g_{ii}$). In evaluating the total energy of the system (6), $E = \int -g^T_{ii}\,\mathrm{d}^3x$, the authors transform this quantity into the surface integral,

$$E = \int g_{ii,j}\,\mathrm{d}S^j. \tag{10}$$

using the coordinate conditions (1) and the fact that the metric becomes asymptotically flat and the coordinate system rectangular at infinity, so that $\mathrm{d}S^j$ is simply the surface element in rectangular coordinates. They then find that the total energy for the point particle is the mass parameter. We have confirmed this physically reasonable result using the expressions (7). However, because of the highly singular and complex behavior of the metric (and consequently the energy density) near the origin, the above replacement of the volume integral by the surface integral is clearly not valid. Indeed, if we include the region $0 \leqslant r < \alpha$ the volume integral does not exist. On the other hand, if we exclude the « non-physical » region $r < \alpha$, the surface integral taken over a sphere of radius α yields a negative contribution to the total energy of amount $-M$, so that there is zero total energy in the region $\alpha \leqslant r \leqslant \infty$. Thus the energy density either vanishes identically in this region or is not positive-definite. Since the terms of order (α/r) give a vanishing energy density, any departures come from the higher order terms in (7). Using these terms, we find that actually the energy density becomes negative for $r < 5\alpha$. (Note that this value of r is at a sufficient distance outside the Schwarzschild radius so that the expansion (7) is valid.)

We see, therefore, that the theory does not have an intrinsically positive-definite energy density built into it, even for regions outside the Schwarzschild singularity. In the quantized case, we may therefore expect additional difficulties to those encountered in the usual field theories when the gravitational field is coupled to point particles. By way of comparison, it is interesting to note that the somewhat analogous and earlier Hamiltonian approach of Dirac (7) (based on giving the g_{ij} a preferred role over the $g_{0\mu}$ in the dynamical description of the gravitational field, *i.e.*, the specification of the state at a given time) also encounters difficulties for a point particle near the origin where the determinant of the g_{ij} ceases to be positive-definite. In contrast, the energy-momentum complex of Møller (8) (in which the role of the $g_{0\mu}$ is predominant in specifying the energy density) yields a vanishing energy density in the space surrounding a static or stationary distribution of matter. So that in comparison with the electromagnetic field, the energy density of the gravitational field behaves very much like a charge density. However, since the energy-momentum complex is not a true tensor, one cannot legitimately draw this conclusion in all coordinate systems. Nevertheless, the result certainly holds for the A.D.M. coordinate system, and consequently there is a basic disagreement

(6) For units in which $16\pi G c^{-4} = 1$, $c = 1$.

(7) P. A. M. Dirac: *Phys. Rev.*, **114**, 924 (1959).

(8) For a detailed discussion of the problems connected with assigning an energy density to the gravitational field see, C. Møller; *Math. Fys. Medd. Dan. Vid. Selsk.*, **31**, no. 14 (1959); *Max Planck Festschrift* (Berlin, 1958), p.139; *Ann. Phys.*, **4**, 347 (1958); see also J. N. Goldberg; *Phys. Rev.*, **111**, 315 (1958).

between the two approaches. Indeed, at the present time, there does not exist a definition of the energy density of the gravitational field for which there is general agreement and which fulfills all the basic requirements one would like to impose on this quantity.

As a supplement to this note, we shall attempt to list these requirements and give a brief discussion of the solution we have found. In formulating these requirements we have insisted on strict covariance, since without this principle, one no longer has any consistent basis for deriving the field equations; moreover, one no longer has the freedom to choose the coordinate system to fit the problem, but instead must fit the problem to the coordinate system, and this, as we have seen in the above example, can lead to unnecessary complications.

We therefore proceed to look for an energy-momentum expression $E_{\mu\nu}$ which should have the following properties: 1) it should be a true tensor; 2) symmetric; 3) satisfy a covariant conservation law; 4) and, if possible, a true conservation law; 5) yield an intrinsically positive-definite energy density $E_0^0 > 0$; 6) act as a source in the gravitational field equations in the same way that matter does (in accordance with the principle of equivalence), *i.e.*, $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -k(T_{\mu\nu} + E_{\mu\nu})$; 7) and, finally, following DIRAC [7], it should represent a « useful integral of the equations of motion ».

The only tensor we have been able to find that meets all the above requirements is the famous cosmological term of Einstein, $\Lambda g_{\mu\nu}$ with $\Lambda > 0$, so that $E_{\mu\nu} = k^{-1} \Lambda g_{\mu\nu}$ [9]. Because $g_\nu^\mu = \delta_\nu^\mu$ we see that E_ν^μ trivially satisfies a true conservation law since it is actually an invariant. However, note that in general the tensor density $E_\nu^\mu \sqrt{-g}$ does not satisfy a true conservation law except in the so-called proper coordinate systems in which $\sqrt{-g}$ is a constant. We shall discuss in detail the results obtained utilizing $E_{\mu\nu}$ as the energy tensor in a later publication. However, the following observations are of interest in connection with the above problem.

First of all, it is clear that when the field equations are supplemented by the cosmological member, there can be no question of utilizing coordinate conditions requiring space to be flat at infinity [10], since the field equations under these circumstances do not possess such solutions, except for non-physical choice of the $T_{\mu\nu}$. It also follows that in the absence of the cosmological term the energy density of the gravitational field vanishes identically. This result is in substantial agreement that found by Møller for the region outside of matter as described above, except with that the result holds more generally. Moreover, it is interesting to note that if we calculate the energy-momentum density using the complex in a coordinate system for which $g_{0\mu} = -\delta_{0\mu}$, we find it vanishes identically. It is to be emphasized that a vanishing energy density does not necessarily imply a vanishing total energy, since the product with an infinite volume need not by a suitable limiting process. And in fact we have found that in the case of the Schwarzschild field as one passes from a finite to an infinite universe (*i.e.* $\Lambda \to 0$), the total energy actually becomes infinite and also there is a finite remainder proportional to the Schwarzschild mass [11].

[9] Note that the usual Einstein energy-momentum pseudo-tensor reduces to Λg_ν^μ in natural coordinates. See for example, R. C. TOLMAN: *Relativity, Thermodynamics and Cosmology* (Oxford, 1934), p. 225; *Phys. Rev.*, **35**, 875 (1930). It follows from general covariance, if this is the tensor describing a physical quantity in one coordinate system, it describes it in all coordinate systems. See also A. S. EDDINGTON: *The Mathematical Theory of Relativity* (Cambridge, 1923), p. 135.

[10] It is to be emphasized that it is not the coordinate conditions themselves that are excluded (provided they yield physically meaningful solutions), but the supplementary boundary conditions.

[11] Compare the result given in Möller [eq. (53) *Ann. Phys.* article] when the energy of the SCHWARZSCHILD field is calculated in polar coordinates using the Einstein pseudo-tensor.

Finally, it also follows that the homogeneous solutions to $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$, of which « gravitational waves » are an example, correspond to a situation of a vanishing gravitational energy-momentum tensor, and are therefore quite unlike the case of electromagnetic radiation ([12]). Even in the case $\Lambda \neq 0$, we have $E_\nu^\mu = k^{-1}\Lambda\delta_\nu^\mu$, so that E_i^0, E_0^i vanish identically. Also we have $E_\mu^\mu = 4k^{-1}\Lambda$; in contrast with the electromagnetic field, for which $T_\mu^\mu = 0$. Because of this latter relation, electromagnetic radiation behaves as a substance which exhibits a pressure $p = u/3$, where u is the energy density, since $T_i^i = -3p$. Hence in the case of a hohlraum, using this relation between pressure and energy density, one can derive by well-known thermodynamical arguments the blackbody radiation law $u = a\theta^4$. On the other hand, since $E_\mu^\mu = 4k^{-1}\Lambda$, the gravitational field behaves as a substance for which $p = -u$, by the same kind of thermodynamical argument it follows that u is independent of temperature; also for the entropy we have $\mathrm{d}S = (p + u)\,\mathrm{d}V/\theta = 0$ ([13]).

Thus, in conclusion, we see that even without incorporating quantum theory explicitly, the gravitational « field » behaves in an extremely non-classical manner. As we shall argue in a subsequent publication apart from numerical factors, $E_0^0 = k^{-1}\Lambda = G\hbar^2/l^6c^2$ (where $l \approx 3 \cdot 10^{-13}$ cm), so that actually the non-vanishing of the cosmological term and hence the energy density of the gravitational field is directly connected with the non-vanishing of Planck's constant. Such an intimate (and at first glance, paradoxical) connection between cosmological quantities on the one hand, and microscopic quantities on the other, as we have pointed out earlier ([14]) is to be expected on the basis of Mach's principle.

* * *

The author wishes to thank Professor C. Møller for some stimulating and critical discussions concerning some of these questions. He would also like to acknowledge the kind hospitality afforded him by Niels Bohr a the Institute for Theoretical Physics in Copenhagen, and also that by Prof. E. R. Caianiello at the Scuola di Perfezionamento in Fisica Teorica e Nucleare in Naples.

([12]) Compare also the results obtained by L. Infeld: *Ann. Phys.*, **6**, 341 (1959), and the extremely small loss of energy due to gravitational radiation obtained by V. Fock: *The Theory of Space Time and Gravitation* (London, 1959), pp. 334-337.

([13]) In a more complete treatment, the problems of defining temperature in a gravitational field would have to be considered, see the discussion in R. C. Tolman and P. Ehrenfest: *Phys. Rev.*, **36**, 1791 (1930).

([14]) *The absolute Lorentz transformation and Mach's principle* (submitted for publication in the *Ann. Phys.*, 1959). As we pointed out there, general relativity, in its present form, does not fully entail Mach's principle and that an extension incorporating quantum theory would be necessary. The road to fulfilling this program, as originally pointed out by Einstein, apparently lies in the incorporation and further interpretation of the cosmological term.

Test of Parity Conservation in Strong Interactions by the Measurement of the β-γ Angular Correlation of ^{133}Xe.

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(ricevuto il 19 Gennaio 1960)

If parity is not conserved strictly, the nuclear states will be mixtures of wave functions with different parities. F is the relative amplitude of the wave function with the wrong parity and delivers a measure for the breakdown of parity ($F=0$, if parity is conserved). Some observations (*e.g.* violation of selection rules, γ-γ-anisotopy with respect to 90°) allow to put an upper limit for F^2. As, however, F is a very small number, much better information can be obtained if F itself instead of F^2 is measured.

This can be achieved by observing the circular polarization P of a γ-radiation emitted from a mixed nuclear state. Neglecting terms of higher powers of F the polarization is given by $P=2RF$, where R is the ratio of the matrix elements for the transitions starting from the wave function with wrong and right parity, respectively. R can be estimated on the basis of nuclear models. The best limit of F obtained so far [1-3] is $F \lesssim 2\cdot 10^{-4}$. Obviously this limit could be pushed down further if transitions with high values of R could be investigated. But it turns out that in all known decays large values of R are coupled with small values of quantum energy thus frustrating an exact measurement of the γ-polarization. Krüger [4] suggested a way to overcome this difficulty. He pointed out that RF can be determined by observing the circular polarization of one γ-ray in a γ-γ-cascade in which any of the two transitions has a high R-value. *E.g.* the polarization of the first transition which may have a large photon energy is measured, whereas the second transition may possess a large R.

A further simplification can be achieved if the first transition is a β-decay. As β-particles are longitudinally polarized due to the parity violation in weak interactions, the observation of the direction of emission of the β-particle gives the same information as the measurement of the polari-

(1) N. Tanner: *Phys. Rev.*, **107**, 1203 (1957).
(2) D. H. Wilkinson: *Phys. Rev.*, **109**, 1603, 1610, 1614 (1958).
(3) R. E. Segel, J. V. Kane and D. H. Wilkinson: *Phil. Mag.*, **3**, 204 (1958).

(4) L. Krüger: private communication by Dr. Kramer and *Zeits. f. Phys.*, **157**, 369 (1959). We would like to thank Dr. Krüger for making available to us his results before publication.

zation of the first γ-ray (5). Therefore an ordinary β-γ-angular correlation measurement would prove a violation of parity conservation if an asymmetry with respect to 90° would be found.

The decay of the 81 keV-level in ^{133}Cs is a γ-transition with a high R-value (4). As the β-decay of ^{133}Xe leads to this level, this β-γ-cascade seemed to be very convenient for such angular correlation measurements.

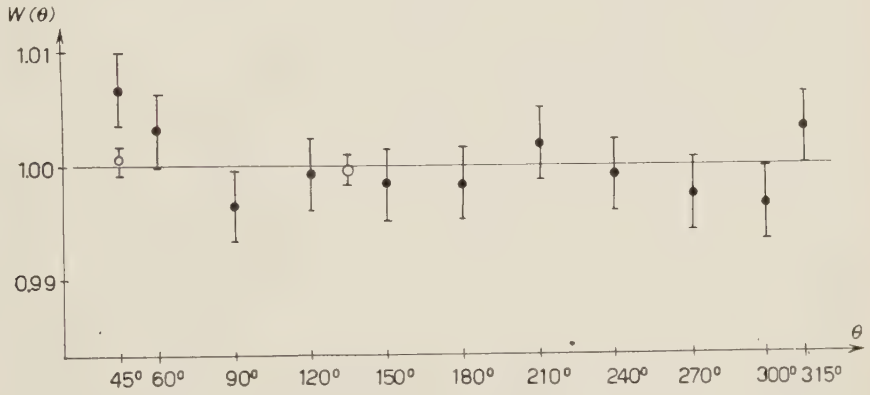

Fig. 1. – β-γ-angular correlation for ^{133}Xe: O run 1; ● run 2.

The radioactive source was a thin glass capillary (0,7 cm long, 0,8 mm wide) filled with Xe-gas at a pressure of about 1 atm. The wall thickness of the capillary determined by weighing was less than 4 mg/cm^2. It is very important to keep the glass wall as thin as possible as a thickness only 2 or 3 times as large as the one used would smear out a small anisotropy completely. The source was mounted inside a plexiglas envelope which was attached to the β-counter. This whole assembly could be rotated around the axis of the source whereas the γ-counter was stationary. The plexiglass envelope was filled with hydrogen gas in order to decrease the scattering of the β-particles. The solid angles were defined by tin cones which also prevented the scattering of the quanta from one counter to the other. The β-particles were detected by a plastic scintillator, the photons by a NaJ-crystal (1½ in. × 1 in.). The photomultipliers were connected to an ordinary slow-fast coincidence circuit. The angle was changed every 2 min and the number of coincidences was divided by the single counting rates. No corrections for the finite size of source and counters were applied.

The results of two runs are shown in the Figure. A least square fit analysis results in $A = (0{,}0012 \pm 0{,}0010)$ where A is defined by $W(\theta) = 1 + A \cos\theta$. In order to investigate the influence of second forbidden matrix elements we looked also for a $P_2(\cos\theta)$-term. The coefficient of such a term can be inferred from run 2 as $(0{,}0017 \pm 0{,}0020)$.

For the β-γ-cascade of ^{133}Xe one obtains $A = \frac{2}{3}RF$. R was estimated by KRÜGER (4). Assuming that the parity-forbidden part of the γ-transition can be calculated by the single particle model he obtained $R = 290$. The other extrem would be that both parts of the γ-transition are equally retarded and it follows

(5) This possibility was suggested to one of the authors by Dr. LIPKIN in connection with the determination of the sign of quadrupol moments.

$R \simeq 12$. From this two extrem values of R and the experimental result for the asymmetry one can deduce the following limits: $F \leqslant 5 \cdot 10^{-6}$ and $1.3 \cdot 10^{-4}$ respectively.

While this investigation was in progress we were informed that similar measurements, though for a smaller range of angles, were performed by F. BOEHM and W. HAUSER at Heidelberg and by E. BODENSTEDT at Hamburg. All these experiments are in very good agreement. We would like to thank these authors for communicating us their results before publication.

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Elastic Scattering of Nucleons and Pions at Very High Energy.

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(ricevuto il 20 Gennaio 1960)

In investigating the high energy (several GeV) elastic scattering of nucleons and pions, two simplifying assumptions have been commonly made (**):

a) The real parts of all the phase shifts vanish.

b) Either the scattering is spin-independent (pion-proton scattering) or the scattering amplitude, which is in general a matrix in spin space, is highly degenerate (proton-proton scattering).

The experiment of Markov *et al.* (2), which is supported by further experiments recently carried out in Dubna (**), contradicts, however, with this simple theory (**). We have analysed the problem of high energy scattering and examined possible ways to get rid of this discrepancy. The present note is meant to give a brief account of the results obtained.

1. – The conditions for the applicability of the simple theory can be summarized as follows:

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(**) A general survey on this problem was given by V. I. Veksler (1). An extensive list of relavant works can be found in it.

(**) We are grateful to Prof. M. Danysz and Dr. P. Zieliński for this information.

(**) This point was emphasized by V. I. Veksler (1). He then describes the introduction of a complex optical potential as a sensible approach.

(1) V. I. Veksler: *Report at the IX Intern. Conf. on High Energy Physics* (Kiev, July 1959).

(2) P. Markov *et al.*: *Report at the IX Intern. Conf. on High Energy Physics* (Kiev. July 1959).

Necessary and sufficient condition

$$a)\ 0 \leqslant \eta_l \leqslant 1 \text{ where } \eta_l = 1 - \int_{-1}^{+1} \left(\frac{\mathrm{d}\sigma_{\mathrm{el}}}{\mathrm{d}\Omega}\right)^{\frac{1}{2}} \mathrm{d}(\cos\theta)\,.$$

Necessary conditions

$$b)\ \sigma_{\mathrm{inel}} \neq \sigma_{\mathrm{el}};$$

$$c)\ \int_{-1}^{+1} \left(\frac{\mathrm{d}\sigma_{\mathrm{el}}}{\mathrm{d}\Omega}\right)^{\frac{1}{2}} \mathrm{d}(\cos\theta) \leqslant \lambda\!\!\!^{-};$$

$$d)\ \left(\frac{\sigma_{\mathrm{tot}}}{4\pi\lambda\!\!\!^{-}}\right)^2 \geqslant \left(\frac{\mathrm{d}\sigma_{\mathrm{el}}}{\mathrm{d}\Omega}\right)_\theta \qquad \text{for arbitrary } \theta.$$

e) No polarization effects are to be observed. (This is true even if we admit spin-dependence).

f) When $\mathrm{d}\sigma_{\mathrm{el}}/\mathrm{d}\Omega = 0$ for $\theta \geqslant \theta_0$, then η_l increases with l at least up to $l \leqslant \pi/\theta_0$;

g) If $\eta_l = 1$ for $l \geqslant l_0$, $\mathrm{d}\sigma_{\mathrm{el}}/\mathrm{d}\Omega$ cannot vanish for $\theta \leqslant \pi(2l_0+1)$.

2. – The optical model has been usually applied to the high energy scattering problem. When $\mathrm{Re}\, V \neq 0$ and moreover its sign is constant (V denoting the optical potential), the above condition *d*) can be generalized into

$$\left(\frac{\sigma_{\mathrm{tot}}}{4\pi\lambda\!\!\!^{-}}\right)^2 \left(1 + \left|\frac{\mathrm{Re}\, f(0)}{\mathrm{Im}\, f(0)}\right|^2\right) \geqslant \left(\frac{\mathrm{d}\sigma_{\mathrm{el}}}{\mathrm{d}\Omega}\right)_\theta,$$

where $f(0)$ stands for the scattering amplitude. (Spin-independence is understood).

When the scattering is spin-independent, we can derive from a set of known phase shifts the functions K and k_1, which specify the complex potential in the optical model theory:

$$K(\varrho^2) = -\frac{4}{\pi} \int_{\varrho^2}^{\infty} \frac{\partial}{\partial y} (\mathrm{Im}\, \delta(y)) \cdot \frac{\mathrm{d}y}{\sqrt{y - \varrho^2}}\,,$$

$$k_1(\varrho^2) = -\frac{2}{\pi} \int_{\varrho^2}^{\infty} \frac{\partial}{\partial y} (\mathrm{Re}\, \delta(y)) \cdot \frac{\mathrm{d}y}{\sqrt{y - \varrho^2}}\,,$$

where $\delta(\varrho^2)$ is the function obtained from the phase shifts by the transition $l + \frac{1}{2} \to \varrho k$. With the experimental data available at present, the phase shift analysis can be carried out in an unambigous way only if we assume $\mathrm{Re}\, \delta_l = 0$.

3. – We have performed a preliminary examination of the possibility of applying dispersion relations in order to estimate the magnitude of the real part of the scattering amplitude for p-p collision. Such a calculation is of interest, since it can

be a direct check of the presence of potential scattering. Before further calculations are carried out we cannot make a definite statement; it seems, however, that at present this approach is not sensitive enough to give a clear-cut conclusion.

4. – We have analysed the implications of admitting spin-dependence of the scattering in the nucleon-nucleon collision. If we assume $\mathrm{Re}\, f_{ij} = 0$ then we have instead of the usual optical theorem its generalisation:

$$\left(\frac{\mathrm{d}\sigma_{\mathrm{el}}^{(\mathrm{exp})}}{\mathrm{d}\Omega}\right)_{\theta=0} = Q(\xi)\left(\frac{\sigma_{\mathrm{tot}}^{(\mathrm{exp})}}{4\pi\lambda\!\!\!^{-}}\right)^2.$$

In the case of neutron-proton scattering we have additionally:

$$\left(\frac{\mathrm{d}\sigma_{\mathrm{el}}^{(\mathrm{exp})}(\mathrm{n\text{-}p})}{\mathrm{d}\Omega}\right)_{\theta=\pi} = Q(\eta)\left(\frac{\sigma_{\mathrm{tot}}^{(\mathrm{exp})}(\mathrm{p\text{-}p}) - \sigma_{\mathrm{tot}}^{(\mathrm{exp})}(\mathrm{n\text{-}p})}{4\pi\lambda\!\!\!^{-}}\right)^2,$$

where

$$Q(x) = \frac{4(4x^2 - 6x + 3)}{(3 - 2x)^2}, \qquad \eta = \frac{\xi\xi'(1-\zeta)}{\xi' - \xi\zeta},$$

the superscripts (exp), (s), (t) denoting respectively the experimentally measured values for unpolarized beam and those corresponding to the singlet and triplet states. Further ξ and ξ' are the ratio $\sigma_{\mathrm{tot}}^{(\mathrm{s})}/(\sigma_{\mathrm{tot}}^{(\mathrm{s})} + \sigma_{\mathrm{tot}}^{(\mathrm{t})})$ for the p-p and p-n case respectively, while ζ stands for the ratio $\sigma_{\mathrm{tot}}^{(\mathrm{s})}(\mathrm{p\text{-}p})/\sigma_{\mathrm{tot}}^{(\mathrm{s})}(\mathrm{n\text{-}p})$. The discrepancy with Markov and others' experiment is removed if the singlet interaction is assumed to predominate strongly over the triplet one (*). Such a situation may imply the following effect.

If the contribution to the proton-neutron scattering from the isotopic singlet part is not very large, the elastic scattering cross section of proton by a nucleon bound in a nucleus as ^{4}He or ^{12}C will be much reduced compared to the free p-p case.

5. – A third possibility is to assume that for certain l's we have

$$\eta_l = \eta_l^{(0)} + \eta_l^{(1)} \cdot \exp[i\xi],$$

$\eta_l^{(1)}$ and ξ changing very rapidly within ΔE, where ΔE means the energy spread of the incident beam. Here again the spin-dependence is neglected. When averaged over ΔE, we obtain

$$\left(\frac{\mathrm{d}\sigma_{\mathrm{el}}}{\mathrm{d}\Omega}\right)_{\theta=0} = \left(\frac{\overline{\sigma_{\mathrm{tot}}}}{4\pi\lambda\!\!\!^{-}}\right)^2 + \sum\sum(2l'+1)(2l''+1)\overline{\eta_{l'}^{(1)}\eta_{l''}^{(1)}}.$$

Thus we can overcome the disagreement of Markov and other's result if we assume a quasi-resonance, which involves at least $2 \div 3$ partial waves with rela-

(*) See the *Note added in proof*.

tively large l. We expect that such a phenomenon, if it really takes place, will be due to the pion-pion interaction.

A more detailed account will be published shortly.

* * *

One of us (Z.K.) express his gratitude to Polish Academy of Sciences, especially to Professor L. INFELD, for the hospitality.

Note added in proof.

A recent work of SHAHBAZYAN, in Dubna, who has carried out a detailed phase shift analysis of MARKOV and others' experiment, assuming spin-dependence but no potencial scattering, has shown that the latter assumption leads to contradiction. This point was not taken into account in our argument, but can be understood in the following way. If we assume the predominance of the singlet state interaction, as discussed in the text, we have at the same time to modify the condition 1.*c*) into a more restrictive form, and it is found very difficult to fit the experimental data to this criterion. (Feb. 27, 1960).

IL NUOVO CIMENTO Vol. XV, N. 5 1° Marzo 1960

Polarization Effects in the Elastic Scattering of Photons.

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(ricevuto il 25 Gennaio 1960)

1. - Introduction.

With the present experiment we have determined quantitavely the polarization effects on the elastic scattering of photons. In a previous paper [1] we outlined that such measurements constitute, together with the elastic scattering of polarized photons, a valid test for an experimental check of the various theories proposed on this argument.

Recently Böbel and Passatore [2] dealing with the same problem have calculated in detail the polarization effects. At the same time Sood [3] measured the degree of linear polarization due to elastic scattering for the energies of 0.411 MeV, 0.662 MeV and 1.25 MeV.

At the last energy and for a scattering angle $\theta=90°$ Sood found a percentage of polarization of $(6\pm2.5)\%$ quite different from the theoretical value of 20% evaluated by Böbel and Passatore and by us in the previous paper, when the interference due to Thomson effect, which is not negligible in this case, is taken into account.

The results of Sood could be considered as fully satisfying for a confirmation of the theory of Brown and co-workers [4-6] on the Rayleigh scattering in comparison with that proposed by Franz [7].

However it is interesting to verify the polarization effects at angles different from 90° where the evaluation of the degree of polarization is less influenced by the numerical approximations and where the experimental results, as quantitative tests, can have a greater significance.

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(1) D. Brini, E. Fuschini, D. S. R. Murty and P. Veronesi: *Nuovo Cimento*, **11**, 533 (1959).

(2) G. Böbel and G. Passatore: XLIV Congresso Nazionale della Società Italiana di Fisica. Riassunto delle Comunicazioni (Palermo, 1958).

(3) B. S. Sood: *Proc. Roy. Soc.*, A **247**, 375 (1958).

(4) S. Brenner, G. E. Brown and J. B. Woodward: *Proc. Roy. Soc.*, A **227**, 59 (1955).

(5) G. E. Brown and D. F. Mayers: *Proc. Roy. Soc.*, A **234**, 387 (1956).

(6) G. E. Brown and D. F. Mayers: *Proc. Roy. Soc.*, A **242**, 89 (1957).

(7) W. Franz: *Zeits. f. Phys.*, **98**, 314 (1936).

2. - Verification of the polarization state.

The polarization state can be verified by measuring the ratio:

$$R' = \frac{d\sigma_0/d\Omega + \xi_{\parallel}(d\sigma_1/d\Omega)}{d\sigma_0/d\Omega + \xi_{\perp}(d\sigma_1/d\Omega)}, \tag{1}$$

with $\xi_{\parallel} = -\xi_{\perp}$, where ξ is the polarization degree of the elastically scattered beam, taking into account the interference between Rayleigh and Thomson effects, and where [8]:

$$\begin{cases} \dfrac{d\sigma_0}{d\Omega} = \dfrac{1}{2} r_0^2 \left(\dfrac{k_2}{k_1}\right)^2 [1 + \cos^2\theta_c + (k_1 - k_2)(1 - \cos\theta_c)] \\ \dfrac{d\sigma_1}{d\Omega} = \dfrac{1}{2} r_0^2 \left(\dfrac{k_2}{k_1}\right)^2 \sin^2\theta_c , \end{cases} \tag{2}$$

θ_c being the Compton scattering angle and k_1 and k_2 the energies of the incident and the scattered photons. Substituting the numerical values in eq. (1) it is found that, for different angles of elastic scattering and for various theories, it exists an angle θ_c for which R' is maximum. The value of θ_c varies with the energy of the γ-rays. For photons of energy $2.56\,mc^2$, θ_c is equal to 70°.

3. - Experiment.

The experiment was performed using a ^{60}Co source of about one curie, emitting photons of a mean energy 1.25 MeV. This energy is slightly less than 1.305 MeV where the calculations of Brown and Mayers were made.

A collimated beam was sent on to a Hg target in the form of a cylinder of 3×3 cm, whose axis was normal to the direction of the beam. A second collimator fixed at an angle of 53°, to which corresponds the maximum polarization degree from the theory of Brown and Mayers defined a polarized beam of photons scattered both elastically and inelastically. A successive Compton scattering analysed the polarization state of the beam. A NaI(Tl) crystal mounted on a photomultiplier was used as a target in the second process. This counter detected the electron while the scattered photon was detected by a second similar counter. A coincidence between the two events, after suitable amplification and discrimination, selected the photons that suffered elastic scattering on Hg. Much attention was paid for the correct discrimination of the energy. The measures were performed by keeping the second counter once in the plane of Rayleigh scattering and once in a plane perpendicular to that. The background was measured by introducing a delay in the coincidence circuit. A possible geometrical asymmetry was determined by replacing the Hg target with a small source of ^{60}Co which provided an unpolarized beam of photons.

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[8] U. FANO: *Journ. Opt. Soc. Am.*, **39**, 859 (1949).

4. - Results and conclusions.

The calculated values for R' at $\theta = 53°$ and $\theta_c = 70°$ are the following:

BROWN and MAYERS . . . . . $R' = 2.26$

FRANZ . . . . . $R' = 1.48$

The result obtained in the present experiment, after applying correction for the asymmetry due to geometry, is:

$$R' = 2.3 \pm 0.1 ,$$

which is in good agreement with the theory of Brown and Mayers.

* * *

We are indebted to Mr. G. BUSACCHI and to Mr. R. VOLTA for their constant help in maintaining the electronic equipment.

One of us (D.S.R.M.) expresses his gratitude to Professor G. PUPPI for a maintenance grant and for the hospitality at the Istituto di Fisica in Bologna.

Muon Capture in ^{12}C (*).

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(ricevuto il 29 Gennaio 1960)

Once Fujii and Primakoff calculated the partial transition rate of the muon capture reaction $\mu^- + {}^{12}C \to {}^{12}B + \nu$, in which the daughter nucleus is found in its ground state ([1,2]). Besides the *unavoidable* assumption for the choice of the nuclear model, such as:

(i) the *j-j* coupling shell model with the harmonic oscillator radial wavefunction adjusted to the observed nuclear size, their calculation was a lot simplified by;

(ii) the omission of the relativistic corrections, namely the interaction terms proportional to the differential operator onto the nuclear wavefunction, and

(iii) the neglect of the interferences of relativistic correction terms with the main term.

We have developed a quite general formalism of the muon capture reaction in close analogy to the theory of relativistic corrections in the β-decay process ([3,4]). We report here the improved results of the Fujii-Primakoff calculation, handled with this new machinery, where the assumptions (ii) and (iii) are removed but the same nuclear model (i) is adopted.

The basic interaction Hamiltonian of the muon capture reaction involves two terms which are not well established at present, namely the pseudoscalar coupling term induced by the virtual pion effects and the magnetic moment term added by the assumption of a conserved vector current. We take the ratio of the strength of the pseudoscalar and axial vector coupling constant, C_P/C_A, as a parameter and found the following partial transition rate ω_μ in units of $10^3\ s^{-1}$:

1) With the magnetic moment

(*) This work is partially supported by the U. S. Atomic Energy Commission and by the Air Force Office of Scientific Research.

(1) H. Primakoff: *Rev. Mod. Phys.*, **31**, 802 (1959).

(2) A. Fujii and H. Primakoff: *Nuovo Cimento*, **12**, 327 (1959).

(3) M. Morita and A. Fujii: *Phys. Rev.*, April, 15 (1959).

(4) M. Morita and R. S. Morita: *Phys. Rev.*, **109**, 2048 (1958).

TABLE I. – *Theoretical and experimental transition rates of the muon capture reaction between the ground states of* ^{12}C *and* ^{12}B.

C_P/C_A	Magnetic moment term	ω_μ in units of 10^3 s^{-1}	
		present work	FUJII and PRIMAKOFF
8	included	7.12	7.86
8	omitted	5.68	6.34
-8	included	10.39	11.80
-8	omitted	8.95	10.25
Experiments		9.05 ± 0.95	(a)
		9.18 ± 0.5	(b)
		6.6 ± 1.1	(c)
		6.8 ± 1.5	(d)
		5.9 ± 1.5	(e)

(a) ARGO, HARRISON, KRUSE and MCGUIRE: *paper* S4 *at Conference on Weak Interactions, Gatlinburg, Tenn.* (1958).
(b) BURGMAN, FISCHER, LEONTIC, LUNDBY, MEUNIER, STROOT and TEJA: *Phys. Rev. Lett.*, **1**, 469 (1958).
(c) FETKOVICH, FIELDS and MCILWAIN: *Bull. Amer. Phys. Soc.*, **4**, 81 (1959).
(d) LOVE, MARDER, NADELHAFT, SIEGEL and TAYLOR: *Bull. Amer. Phys. Soc.*, **4**, 81 (1959).
(e) T. N. K. GODFREY: *Ph. D. Thesis* (Princeton, 1954) and *Phys. Rev.*, **92**, 512 (1953).

term,

$$\omega_\mu = 6.28 + 0.00481 \cdot [(C_P/C_A) - 21.2]^2 ,$$

The results are summarized in Table I and plotted in the Fig. 1. For the most accepted value of $C_P/C_A \sim 8$(5), our

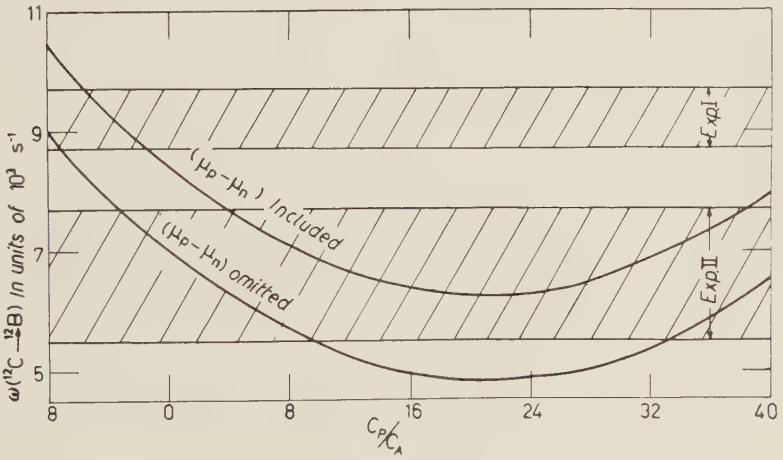

Fig. 1.

2) Without the magnetic moment term,

$$\omega_\mu = 4.84 + 0.00481 \cdot [(C_P/C_A) - 21.2]^2 .$$

(5) M. L. GOLDBERGER and S. B. TREIMAN: *Phys. Rev.*, **111**, 354 (1958).

prediction is about 10% less than that of Fujii and Primakoff. We notice that the primary question of the existence of the magnetic moment term is in fact entangled with the parametric value of C_P/C_A which is not well known up to date. Moreover, the ambiguities involved in the nuclear model (i) ([6,7]), the finite size correction and the relativistic correction for the muon wavefunction ([8]) may shield the delicate point of the primary question. Hence the further study of the subject is desirable.

* * *

The authors thank Mr. I-TUNG WANG of Columbia University for assisting the numerical computation.

([6]) The transition rate of the β decay $^{12}B \rightarrow {}^{12}C + e + \tilde{\nu}$ is calculated to be $\omega_\beta = 159\ s^{-1}$ with the nuclear model (i). This is 4.8 times the observed experimental rate; but for the ratio ω_μ/ω_β the error is supposed to be fairly small though the quantitative estimate of the error is very difficult.

([7]) L. WOLFENSTEIN: *Nuovo Cimento*, **13**, 319 (1959).

([8]) The last two corrections are of the order of αZ, which is about 4.4% for ^{12}C. Recently FLAMEND and FORD: *Phys. Rev.*, **116**, 1591 (1959) showed that they are in fact as much as −6% in ^{12}C.

Experimental Evidence for the Pion-Pion Interaction at 1 GeV.

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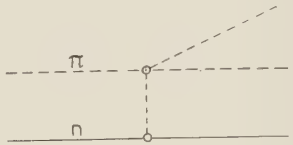
(ricevuto il 1° Febbraio 1960)

The experimental results so far obtained on the production of single charged pions in $\pi+p$ collisions can be explained quantitatively if one assumes that the process proceeds predominantly via the $(\frac{3}{2}, \frac{3}{2})$ isobar. This predominance results from the existence of the pure $T=\frac{3}{2}$ (π^-n) state among the outgoing particles in this process. However, the production of neutral pions is not satisfactorily explained by this assumption [1].

For the production of a neutral pion the pure $T=\frac{3}{2}$ state does not exist, and it is reasonable to expect important contributions from other possible interactions (for example $\pi\pi$). For this reason it should be easier to measure the effects of $\pi\pi$ interaction in the production of neutral pions. From this point of view the recent paper by Bonsignori and Selleri (BS) [2] is particularly interesting. They make theoretical predictions concerning π production on the assumption of a $\pi\pi$ interaction. Thus, in a pion-nucleon collision, the incident pion is able to produce a second pion from the virtual meson cloud.

Assuming that the production process is that shown in the accompanying Feynman diagram:



new line.

they are able to predict the existence of a low energy peak in the spectrum of the protons in the laboratory system for the reaction:

$$\pi^- + p \to \pi^- + \pi^0 + p \,.$$

We have worked out this spectrum for π^--mesons of 1 GeV on the basis of these assumptions and also on the basis of the statistical model. (N.B. - For the Lindenbaum-Sternheimer model [3] one obtains almost the same curve as for the statistical model if one assumes that the isobar is emitted isotropically in the c.m.s. of the colliding particles.) These theoretical curves are now com-

(1) I. Derado and N. Schmitz: to be published in *Phys. Rev.*

(2) F. Bonsignori and F. Selleri: *Nuovo Cimento*, **15**, 465 (1960).

(3) S. J. Lindembaum and R. M. Sternheimer: *Phys. Rev.*, **109**, 1723 (1958).

pared with the experimental results already published (now with improved statistics) (4).

The histogram in Fig. 1 represents our experimental measurements with the statistical errors. Curve A represents the predictions of the statistical model and curve B those of the (BS) model.

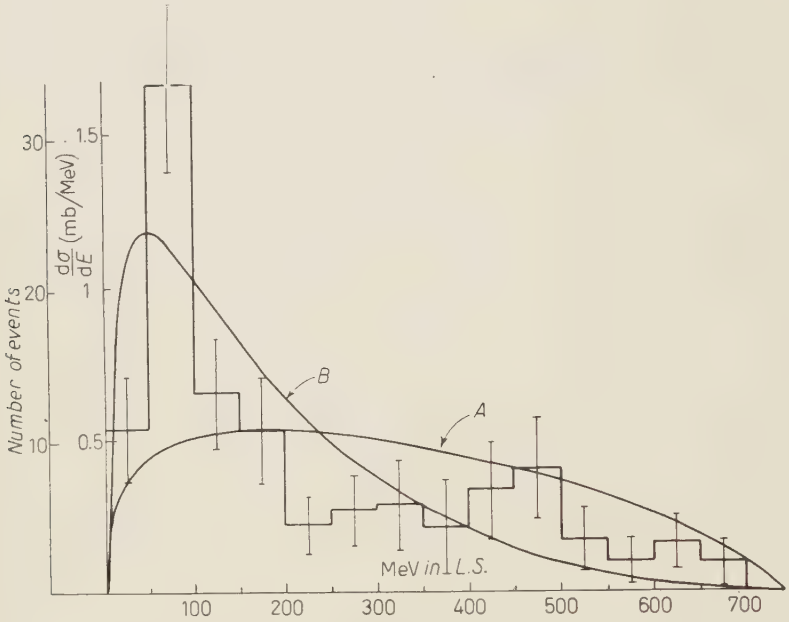


Fig. 1.

The peak at 50 MeV can thus be interpreted as the effect of a $\pi\pi$ interaction. It is evident that curve B is in much better agreement with experiment than curve A.

By normalizing curve B to the area of the histogram, the mean value of the $\pi^- + \pi^0$ scattering cross-section is estimated to be

$$\bar{\sigma} \simeq 30 \text{ mb} .$$

It must, however, be realized that this gives only the order of magnitude of the pion-pion cross-section.

If a $\pi\pi$ resonant state ($\pi\pi$) exists we would expect the reaction to proceed thus:

$$\pi^- + p \to p + \pi\pi \to p + \pi^- + \pi^0 ,$$

so that one would expect to find a strong peak in the spectrum of recoil protons in the c.m.s. If we take into

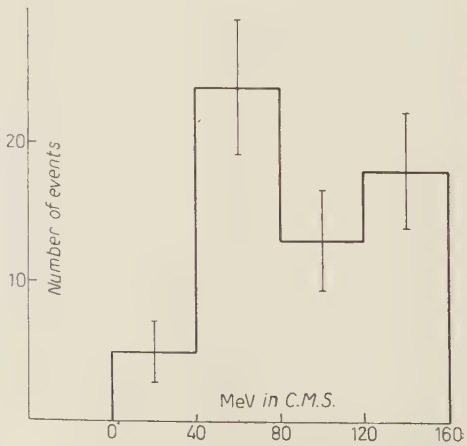


Fig. 2.

(4) I. DERADO, G. LÜTJENS and N. SCHMITZ: *Annal. der Phys.*, **4**, 103 (1959).

account only these protons in the neighbourhood of the maximum in Fig. 1, the effect of the resonance, if it exists, should be all the more apparent. Thus, the histogram shown in Fig. 2 represents the c.m.s. energy distribution of the protons occurring in the interval (0÷190) MeV in Fig. 1. This result suggests that the maximum may occur in the neighbourhood of 60 MeV in the c.m.s. If one assumes this maximum to be real, then one can calculate the energy of the resonant $\pi\pi$ state:

$$m_{\pi\pi} = 4.7 m_\pi ,$$

where m_π is the pion mass.

From this results it appears that the $\pi\pi$ interaction is important in the production process. By a combination of $(\frac{3}{2}, \frac{3}{2})$ isobar and the assumption of a $\pi\pi$ interaction it is perhaps possible to explain the present experimental results on π production. The present experimental results suggest that the $\pi\pi$ interaction becomes increasingly important above 1 GeV.

* * *

We are grateful to Dr. Y. Goldschmidt-Clermont and Dr. Selleri for numerous valuable discussions. We are also indebted to Dr. D. W. Powell and Dr. R. Van de Walle for their helpful interest in this paper.

Derivation of the Classical Grand Canonical Distribution.

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(ricevuto il 3 Febbraio 1960)

At the present the most satisfactory [1] approach to ensemble theory in classical statistical mechanics appears to be Khinchin's [2,3]. The Gibbs canonical ensemble is set up, in which N systems identical with the one system of interest are in weak thermal contact, forming a heat bath with total energy NU. It is first proved that for a phase function referring to one system only the microcanonical average over the whole phase space of the ensemble is equal to the canonical average over the phase space of the system, with error $0(1/N)$. For a canonical ensemble the Hamiltonian takes the form $H=\sum_{i=1}^{N} h(x_i)$, where the symbol x_i represents all the phase variables of the i-th system. The microcanonical average of a function $f(x)$ depending only on the first system is

$$\int f(x_1)\,\varrho(x_1)\,\mathrm{d}x_1\,,$$

where

$$\varrho(x_1) \equiv \varrho(h(x_1)) = C^{-1}\int \delta\Big(NU - \sum_{i=1}^{N} h(x_i)\Big)\,\mathrm{d}x_2 \ldots \mathrm{d}x_n\,,$$

and C is a normalizing constant such that

$$\int \varrho(x_1)\,\mathrm{d}x_1 = 1\,.$$

The phase integrals are estimated with the aid of the central limit theorem of probability theory, and the canonical distribution is obtained,

$$\varrho = Z_\beta^{-1} \exp\left[-h(x_1)\right] + 0(1/N)\,,$$

(1) P. Caldirola: *Nuovo Cimento*, **14**, 260 (1959).
(2) A. I. Khinchin: *Mathematical Foundations of Quantum Mechanics* (New York, 1949).
(3) C. Truesdell and D. Morgenstern: *Erg. ex. Naturwiss.*, **30**, 286 (1958).

where

$$Z_\beta = \int \exp\left[-h(x_1)\right] \mathrm{d}x_1 ,$$

and β is the root (which can be shown to exist and to be unique) of

$$U = -(\partial/\partial\beta) \log Z_\beta .$$

The purpose of this letter is to point out that the grand canonical distribution function can be obtained in the same way, including the Tetrode denominator $n!$ (which is difficult to derive from simpler arguments about the independence of subsystems). The grand canonical ensemble consists of Nn particles distributed over N imperfect enclosures of volume V, and the Hamiltonian H is such that a phase of the ensemble with a large fraction of the particles outside the enclosures has high energy (that is, we assume a high work function for the enclosures) and so is microcanonically unlikely. By neglecting such phases (which is analogous to neglecting phases of the canonical ensemble in which much energy is in the interaction mechanism), we restrict the configuration variables in the phase integrals for the ensemble to the N volumes V; the coordinates of each particle run over a total volume NV.

For those ensemble phases in which there are n_i particles in the i-th enclosure (with $\sum_{i=1}^{N} n_i = Nn$), the Hamiltonian H takes the form $\sum_{i=1}^{N} h(n_i, x_i)$, where $h(n_i, x_i)$ is the Hamiltonian for n_i particles confined to a volume V, and the symbol x_i denotes the phase variables of these particles. An ensemble phase integral is the sum of $(Nn)^N$ terms, in any one of which each of the Nn particles is confined to one of the N enclosures of volume V. There will be $(Nn)!\,[n_1!\,n_2!\,\ldots\,n_N!]^{-1}$ terms with any given set of n_i, and such terms will be equal. We evaluate the microcanonical average of functions referring to the first enclosure only; that is, we have a sequence of functions $f(n, x)$, where the symbol x denotes the phase variables of n particles, and evaluate

$$\bar{f} = C^{-1} \sum_{n_1} \ldots \sum_{n_N} \delta\Big(Nn - \sum_{i=1}^{N} n_i\Big)(Nn)!\,\big[n_1!\,\ldots\,n_N!\big]^{-1} \int f(n_1, x_1)\, \delta\Big(NU - \sum_{i=1}^{N} h(n_i, x_i)\Big)\, \mathrm{d}x_1 \ldots \mathrm{d}x_N ,$$

where $\int \mathrm{d}x_1 \ldots \mathrm{d}x_N$ denotes an integral over the phase space of all the particles, the configurations variables running over a volume V. C is a normalizing constant such that $\bar{1} = 1$. By KHINCHIN's method, we obtain

$$\bar{f} = \sum_n (Z_{\beta\gamma}\, n!)^{-1} \int f(n, x)\, \exp\left[\gamma n - \beta h(n, x)\right] \mathrm{d}x + 0(1/N) ,$$

on estimating numerator and denominator by means of the central limit theorem, in a form applicable to the bivariate distribution

$$(Z_{\beta\gamma}\, n!)^{-1} \exp\left[\gamma n - \beta\varepsilon\right] \sigma(n, \varepsilon) ,$$

in discrete n and continuous ε. $Z_{\beta\gamma}$ is the normalizing integral

$$\sum_n (n!)^{-1} \int \exp\left[\gamma n - \beta\varepsilon\right] \sigma(n, \varepsilon)\, \mathrm{d}\varepsilon \qquad \text{and} \qquad \sigma(n, \varepsilon) = \int \delta\big(\varepsilon - h(n, x)\big)\, \mathrm{d}x\,,$$

for a single system. β and γ are determined from the equations

$$U = -(\partial/\partial\beta) \log Z_{\beta\gamma} \qquad \text{and} \qquad n = (\partial/\partial\gamma) \log Z_{\beta\gamma}\,.$$

It can be shown by considerating the convex function

$$\log\, Z_{\beta\gamma} + \beta U - \gamma n\,,$$

that there is one and only one solution in β, γ of these equations for $0 < \beta < \infty$, $-\infty < \gamma < \infty$.

It can then be shown with only trivial modifications of the known proof [3] for the canonical ensemble that for a function of the form $N^{-1} \sum_{i=1}^{N} f(n_i, x_i)$ the time average is almost everywhere almost equal to the microcanonical average.

LIBRI RICEVUTI E RECENSIONI

R. A. Smith – *Semiconductors*. Cambridge University Press, 1959, pp. xvii-494.

Questa opera, per chiarezza e vastità di trattazione, sembra destinata ad avere una importanza particolare nella letteratura fiorita in questi ultimi anni intorno ai problemi della fisica dei semiconduttori. Nelle intenzioni dell'Autore essa è scritta soprattutto per i fisici e per quegli ingegneri impegnati nello studio e nello sviluppo dei componenti elettronici a semiconduttori.

Tutta la materia è divisa in dodici capitoli. Nei primi otto, che comprendono circa 320 pagine, viene svolta la teoria fisica dei materiali semiconduttori. La trattazione è basata sul concetto di « massa efficace » e si fa ricorso alla teoria quantistica dei solidi in forma elementare solo per illustrare il moto degli elettroni nei cristalli. Anche gli sviluppi matematici sono sempre limitati ai metodi generalmente noti nel primo biennio di studi universitari. Per il carattere di semplicità il libro può essere opportunamente consigliato anche come testo di un corso sui semiconduttori per studenti.

I metodi di misura delle caratteristiche fisiche dei semiconduttori vengono descritte nel nono capitolo, mentre nel decimo e undicesimo sono passate in rassegna le proprietà degli elementi e dei composti semiconduttori. Infine nell'ultimo capitolo si trova una succinta rassegna delle numerose applicazioni avute nei più svariati campi della tecnica.

Il libro, oltre ad essere consigliabile come testo di studio per il suo stile chiaro e di facile lettura, si presta anche ad essere usato come testo di consultazione per i numerosi dati contenuti specialmente negli ultimi quattro capitoli.

V. Pellegrini

P. A. M. Dirac – *I principi della meccanica quantistica*. Boringhieri, 1959, pp. 433-xiv, prezzo L. 4 000.

L'importanza del libro di Dirac nella storia della fisica è stata giustamente paragonata a quella dei *Principia* di Newton. Il parallelo fra le due opere non è casuale, nè l'autore, successore di Newton sulla sua stessa cattedra alla Università di Cambridge, ha voluto evitare il confronto. Come il libro di Newton ha dato una sistemazione alla meccanica classica durata per secoli, così si può dire che alla formulazione della meccanica quantistica data da Dirac non c'è altro da aggiungere. Il libro di Dirac è ormai un classico che tutti considerano insuperabile nella impostazione dei concetti fondamentali e nella formulazione astratta dei principi generali. Alcuni tuttavia ritengono che altri testi possano essere più utili come manuali di consultazione per il ricercatore che studia un problema pratico e vuole trovare già svolta o impostata la trattazione del caso particolare che lo interessa, oppure più facili dal punto di vista didattico. La

mia esperienza è invece che nessun altro libro eguaglia questo non solo per valore formativo, ma anche, una volta che lo si conosca a fondo, come testo di riferimento per trovare la risposta alle domande, ai dubbi e ai problemi che chiunque adopera la meccanica quantistica come mezzo di ricerca, si trova spesso dinnanzi.

È quindi un avvenimento importante per la fisica italiana questa traduzione. Essa permetterà a molti studenti di porre, con questo libro, salde basi alla propria cultura, e a molti studiosi di trovarsi a portata di mano un prezioso strumento di ricerca.

M. Cini

J. Barriol – *Les moments dipolaires.* Gauthier-Villars Ed., Paris, 1957. pp. 184, Tabelle 19, Figg. 67.

Recentemente sono stati scritti parecchi libri sulle proprietà dei dielettrici; la maggior parte però di questi lavori sono molto specializzati e riguardano solo alcuni aspetti particolari del problema. Il libro di J. Barriol ci offre invece uno specchio assai completo di tutti i problemi che riguardano i corpi dielettrici e delle varie soluzioni che sono state proposte.

I problemi sono presentati in modo chiaro e semplice e vengono affrontati con criteri che non postulano una profonda conoscenza della materia, in quanto la loro trattazione prende le mosse dalle definizioni classiche di momento elettrico e di molecole polari e non polari. I concetti teorici, sviluppati parallelamente in modo classico e quantistico, vengono sempre collegati direttamente ai risultati sperimentali, spesso con l'aiuto di utili modelli che permettono una visualizzazione più immediata del problema.

Questo libro, in accordo con le intenzioni dell'Autore, fornisce quindi una guida autosufficiente allo studioso che intenda affrontare questo campo della fisica.

Dopo l'introduzione dei modelli classici della polarizzabilità molecolare, il problema viene rianalizzato con l'apporto della meccanica quantistica e questa trattazione fornisce dei criteri per valutare i limiti di validità della formula di Clausius-Mossotti. Viene poi affrontato il problema dell'assorbimento dielettrico in relazione anche alla interazione tra il momento e la natura del legame, il che permette di ottenere una utile determinazione della struttura delle molecole complesse e dei loro momenti dipolari. Infine vengono esposti con sufficiente chiarezza i più recenti metodi che permettono, almeno in linea di principio, di misurare direttamente i momenti elettrici delle molecole e la loro polarizzabilità sia negli stati eccitati che in quelli fondamentali.

Una cospicua e aggiornata bibliografia permette inoltre al lettore di approfondire maggiormente quelle parti che sono per lui di più immediato interesse.

G. Bonera

T. S. Moss – *Optical Properties of Semi - Conductors.* Butterworths scientific publication. London, 1959, pp. x-279.

Si sono avute finora diverse opere di rassegna generale riguardanti gli studi sui semiconduttori. Attualmente il campo è diventato così vasto da giustificare la compilazione di libri che si occupano solo di determinti aspetti di detti studi: il libro segnalato è fra questi ultimi.

Nei capitoli introduttivi viene sviluppata la teoria delle proprietà ottiche dei solidi, soffermandosi in particolare sui fenomeni di assorbimento e di emissione e sugli effetti magneto-ottici e foto-elettrici nei semiconduttori. Per quanto riguarda invece i processi di elettroluminescenza, eccezionalmente in-

teressanti per numerose applicazioni pratiche, l'Autore ne accenna appena e, poichè questi fenomeni riguardano piuttosto i fosfori che i semiconduttori, rimanda per i dettagli alle ottime monografie di Destriau e Ivey (1955) e alle più recenti di Piper e Williams (1958).

Quasi i due terzi del libro, dieci capitoli su sedici, sono dedicati a una dettagliata elencazione di dati numerici delle caratteristiche elettriche ed ottiche dei più interessanti elementi dei gruppi IV e VI e dei composti intermetallici dei gruppi III e V. Alcuni elementi più interessanti nelle applicazioni, come il Boro, il Silicio, il Germanio, il Selenio ed il Tellurio, occupano ognuno un intero capitolo con una ricca documentazione di tabelle e di grafici che vengono minutamente interpretati e discussi. I dati riportati sono aggiornati ai risultati resi noti nella Conferenza Internazionale sui semiconduttori tenutasi a Rochester nell'Agosto del 1958.

Seguono tre appendici in cui sono ricavate le relazioni che legano fra loro le varie costanti ottiche dei materiali: parte reale e immaginaria della costante dielettrica, ampiezza e fase del coefficiente di riflessione, l'indice di rifrazione e le costanti di assorbimento.

Il fisico sperimentale interessato in queste ricerche troverà nel libro ampio materiale di consultazione nonchè una scelta bibliografia con circa 600 riferimenti.

U. Pellegrini

K. R. Atkins – *Liquid Helium*. Cambridge University Press, 1959 pp. x-309, prezzo 60 s.

Dopo il libro di Keesom non era più comparsa alcuna estesa rassegna dello stato delle ricerche sull'elio liquido; Atkins ha colmato questa lacuna mettendo a disposizione del ricercatore una esposizione, per quanto possibile completa, dei progressi, sia teorici che sperimentali, compiuti negli ultimi venti anni circa la conoscenza del comportamento di questa sostanza e nella soluzione dei problemi generali di meccanica statistica e quantistica da esso sollevati.

Dopo una breve introduzione nella quale vengono esposti sommariamente gli « effetti » fondamentali dell'elio liquido, l'autore passa ad illustrarne, essenzialmente alla luce della termodinamica, l'equazione di stato. Nel terzo capitolo vengono riassunte le teorie base: dalla condensazione di Bose-Einstein, alla teoria delle eccitazioni fondamentali (fononi e rotoni) fino al tentativo di soluzione completa del problema quantistico di molte particelle interagenti. I capitoli successivi sono dedicati ad una esposizione alquanto esauriente dei risultati sperimentali sulle singole proprietà e fenomeni e i tentativi, purtroppo non frequentemente coronati da successo, di interpretazione teorica di essi. Nel quarto capitolo vengono esposti gli aspetti principali della termo-idro-dinamica dell'elio liquido: il flusso attraverso capillari, l'effetto termomeccanico, i tentativi di comprensione della superfluidità. Il quinto capitolo riguarda il primo e secondo suono e la loro attenuazione, il sesto discute a fondo le forze dissipative che si introducono per spiegare il cessare della superfluidità quando viene superata la velocità limite, il settimo descrive il comportamento del film d'elio liquido. Gli ultimi due capitoli infine sono dedicati all'isotopo leggero ^3He, alle sue proprietà nettamente differenti da quelle del ^4He, anche per il fatto che il ^3He ubbidisce alla statistica di Fermi-Dirac, e al comportamento delle miscele di ^3He ed ^4He con speciale risalto a come la presenza degli atomi di ^3He influisce sulle più tipiche proprietà del ^4He.

Data la grande mole di lavori originali riferiti e discussi, il libro rassomiglia più ad un vasto ed esauriente articolo di rassegna che ad una esposizione metodica ed originale della materia; del resto, allo

stato attuale, una esposizione sistematica sarebbe quasi impossibile.

Nel riferire risultati sperimentali l'attenzione è sempre rivolta ad individuare i limiti di essi o gli eventuali errori sistematici, quasi mai alla tecnica impiegata; nell'illustrare risultati teorici il contenuto fisico è sempre a fuoco mentre vengono sempre trascurati gli sviluppi matematici. Queste doti e lo stile vivace e rapido rendono la lettura attraente per il ricercatore già addentro nell'argomento, ma probabilmente un po' ostica per chi ne è completamente digiuno.

G. C. Moneti

G. K. White – *Experimental Techniques in Low-Temperature Physics*. Oxford, at the Clarendon Press, 1959, pp. VI-328, prezzo 45 s.

Come l'autore nota nella prefazione, non esisteva alcun libro che potesse servire da introduzione tecnica al giovane ricercatore che volesse dedicarsi al campo delle bassissime temperature senza avere la possibilità di un lungo periodo di addestramento in un centro di tali ricerche.

Il volume in esame serve eccellentemente allo scopo; l'autore non si preoccupa di discutere a fondo ogni argomento, ma si limita a cenni introduttivi, notizie pratiche ed esempi, accompagnando tutto con molti e precisi riferimenti bibliografici. Questo schema espositivo fa sì che il lettore trovi, per molti semplici problemi di laboratorio, la soluzione direttamente nella consultazione del libro e, per i problemi più particolari, una introduzione alla bibliografia utile.

La prima parte del libro illustra i principi di liquefazione dei gas con la descrizione dei principali tipi di liquefattori in uso, discute i problemi di immagazzinamento e trasferimento dei liquidi freddi e i metodi di misura delle temperature. La seconda parte vuol servire da introduzione alla costruzione del criostato di ricerca; in essa sono illustrati i principali tipi di criostati usati, compresi quelli per irraggiamenti a radiazioni nucleari e raggi X, e quelli per esperimenti con raffreddamento per demagnetizzazione adiabatica. Una discussione sui problemi di scambio di calore (e quindi di isolamento termico) e di controllo della temperatura, e una introduzione alla tecnica ed ai materiali da vuoto completano questa parte.

Chiude il libro una esposizione commentata sui dati disponibili sui calori specifici, i coefficienti di espansione e la conducibilità termica ed elettrica.

Il principale pregio di questo libro sta nella sua estrema praticità: il ricercatore potrà scorrerlo in pochissimo tempo per prendere visione del contenuto; ricorrendo poi ad esso per consultazione egli potrà risparmiare molte noiose ore di perdita di tempo per quesioni tecniche.

Peccato che il ritardo di un anno e mezzo tra la fine della compilazione e la pubblicazione abbia fatto sì che nel libro non siano stati inclusi alcuni importanti sviluppi tecnici recenti.

G. C. Moneti

W. M. Elsasser – *The Physical Foundation of Biology. An analical Study*. Pergamon Press, London, 1958, pp. X-219, prezzo 30 s.

Solo da pochi anni le ricerche biologiche hanno visto scendere al proprio fianco nel tentativo di risolvere in comune i suoi complessi quesiti, la giovane scienza della Cibernetica con il suo fondamento matematico, la Teoria della Informazione.

In realtà la Cibernetica ha già rivelato le proprie possibilità in un numero

sorprendente di campi della ricerca scientifica e dell'ingegneria, ma i problemi implicati nella ricerca biologica, quali ad es. la conservazione e la trasmissione dei caratteri nelle cellule germinali, le correlazioni dei processi chimico-fisici che avvengono nei tessuti, la struttura e la fisiologia del sistema nervoso dal punto di vista propriamente cibernetico, il meccanismo dell'apprendimento e della memoria, sono apparsi subito come i problemi più ardui che dovesse affrontare.

Il libro del Prof. W. M. Elsasser inizia con un'esposizione storica dei problemi e delle correnti di pensiero della biologia, indi si rifà alla competizione sorta nel diciottesimo secolo tra preformazionisti (i quali sostenevano che l'enorme quantità di caratteri di un organismo vivente è completamente contenuta nella cellula germinale originaria) ed epigenetisti (secondo i quali una gran parte di tali caratteri si determina durante lo sviluppo) per definire gli scopi del presente studio nella « giustificazione dal punto di vista epigenetico sulle basi dei metodi e dei concetti della moderna fisica teorica ».

Il contenuto totale d'informazioni immagazzinato nei cromosomi come in una memoria, la « lettura » di una tale memoria da parte dello sperimentatore, la stabilità dei caratteri durante lo sviluppo dell'organismo, ecc., non possono essere interpretati applicando la teoria dell'informazione allo stesso modo che ai sistemi macroscopici creati in laboratorio, se non si vuole giungere a contraddizioni simili a quelle che hanno determinato l'avvento della meccanica quantistica.

Questa tesi è sviluppata attraverso tutto il libro e si appoggia di volta in volta ai concetti cardinali della Cibernetica: nel 1° capitolo alla « reazione », nel 2° alla Teoria dell'Informazione e nel 3° alla conservazione delle informazioni e ai dispositivi di memoria.

Il libro può essere di grande utilità sia per coloro che già hanno familiarità con la Cibernetica e che desiderano conoscere i problemi affrontati con essa nel campo della biologia, sia per gli studiosi di biologia che vi trovano anche un efficace compendio della Cibernetica.

C. Pedicino

L. Eisembud and E. P. Wigner – *Nuclear Structure* – Princeton University Press (Princeton, N. J., U.S.A.) 1958. pag. 128.

Questo volumetto presenta in modo veramente attraente le principali idee direttrici che si sono fin'ora sviluppate nello studio della fisica nucleare. In questo campo, ancora così poco sistemato, i progressi sono stati realizzati mediante lo studio di modelli più o meno adatti a spiegare alcuni fenomeni ed a chiarire, in questo modo, certe caratteristiche particolari dei nuclei. È un indubbio pregio di questo libro la trattazione sintetica di questa sistematica modellistica mettendo in rilievo in modo conciso le loro caratteristiche principali.

Il libro inizia esponendo brevemente le proprietà fenomenologiche dei nuclei: sistematica dei nuclei stabili, caratteristiche degli stati nucleari e reazioni nucleari. Discute dopo il sistema di due nucleoni, i modelli nucleari più interessanti (uniforme, a particelle indipendenti, collettivo) ed i metodi per analizzare alcune proprietà delle reazioni nucleari (risonanze e formula di Breit e Wigner, fluttuazioni delle sezioni d'urto e modello di sfera opaca, modello di risonanza gigante, stripping, ecc.). Susseguentemente sono discusse brevemente le transizioni elettromagnetiche ed il decadimento β. L'analisi di quest'ultimo processo è ancora quella basata sull'ipotesi di conservazione di parità: essa è stata ormai superata dai recenti sviluppi della teoria delle interazioni deboli. Infatti, in questi ultimi due anni il pano-

rama nello studio del decadimento β è radicalmente cambiato grazie al successo ottenuto dalla teoria a due componenti del neutrino e dalla teoria di interazione universale in V-A con non rinormalizzazione della parte vettoriale. Originale e utile il sunto dei riferimenti alla fine del libro: per ogni problema sintetizza i lavori originali o riassuntivi ai quali è più conveniente rivolgersi per un maggiore approfondimento.

La presentazione chiara e sintetica degli argomenti fanno di questo conciso libro un utilissimo strumento per studenti avanzati o per fisici desiderosi di avere un'idea d'insieme in questo campo.

D. Amati

Comptes Rendus du Congrés International de Physique Nucléaire presentés par M.me P. Gugenberger. Paris, Dunod, pag. XXIV-974.

La conferenza internazionale di Fisica Nucleare del luglio scorso a Parigi fu dedicata alle interazioni nucleari di bassa energia ed alla struttura del nucleo. Questo volume della Dunod contiene i testi delle comunicazioni presentate alla conferenza. La prima parte del volume riporta in versione integrale i testi delle esposizioni dei relatori. La lettura di questa prima parte, può risultare utile anche al lettore non specializzato. I soggetti trattati dai vari relatori sono: diffusione elastica, dimensioni e forze nucleari, interazione diretta, reazioni da ioni pesanti, reazioni fotonucleari, modello a particelle indipendente e sue reazioni col modello collettivo, modello collettivo, struttura della materia nucleare ed interazioni deboli. Come si vede, si tratta in complesso di un quadro abbastanza completo degli argomenti di fisica nucleare di bassa energia di recente sviluppo. Occorre dire che non viene presentato molto di nuovo e tantomeno di rivoluzionario in questi campi. Tuttavia appare dalla lettura di questi sommarii come nuove impostazioni vadano ora delineandosi di problemi già da tempo allo studio, come quelli connessi coi modelli ottici, con la teoria generale delle reazioni nucleari, con i modelli collettivi e con la ricerca di una giustificazione del modello a shell.

R. Gatto

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